A. Teleiko, M. Zarichnyi

Categorical Topology of Compact Hausdorff Spaces

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Introduction

Мудрець же физику провадив
І толковав якихсь монадів
І думав, відкіль взявся світ?

І. Котляревський, "Енеїда"

Different constructions of general and algebraic topology demonstrate their categorical nature. The examples include function spaces, products, hyperspaces etc.

The foundation of theory of normal functor in the category of compact Hausdorff spaces have been laid by Evgenii Shchepin. The notion of normal functor turned out to be sufficiently wide to contain many interesting functors and, at the same time, sufficiently special for developing a meaningful theory. Different properties of functors close to being normal were investigated by V. Fedorchuk, A. Dranishnikov, A. Chigogidze, A. Ivanov, M. Smurov, V. Basmanov, E. Moiseev, A. Savchenko, T. Banakh, T. Radul, O. Nykyforchyn, and other authors.

The book is organized as follows. Chapter 1 contains some information from general topology and category theory. The material in Chapter 2 devoted to of general theory of functors in the category of compact Hausdorff spaces and related categories is concentrated around the notion of normal functor. One of the important general problems considered in this chapter is the problem of intrinsic characterization of concrete functors or classes of functors.

Chapter 3 deals with monads generated by functors close to being normal. In particular, we consider here the problem of characterization of the categories of algebras, exiension of functors onto the Kleisli categories, and lifting of functors onto the categories of algebras. The Kleisli categories naturally appear in the topology in the context of multivalued

maps.

In Chapter 4 we consider some applications of functors and monads to geometric topology and in Chapter 5 to general topology of compact nonmetrizable spaces. The exposition in these chapters is necessarily far from being self-contained; the primary objective is to give diverse examples of connections between the theory developed in Chapters 2, 3 and topology of absolute extensors, (infinite-dimensional) manifolds, equivariant topology etc.

The book considerably extends the monograph by the second-named author (M. Zarichnyi [1993]) published in Ukrainian. We would like to express our gratitude to Professor R. Cauty for correcting some errors in this monograph.

Last but not least: the authors are indebted to Taras Radul who directed their attention to the place in the immortal poem "Eneida" by Ivan Kotlyarevs'kyi, where the monads were mentioned.

Andrii Teleiko
Michael Zarichnyi

Chapter 1.

Preliminaries

1.1. General Topology

The term "space" usually means "topological space", "map" means "continuous map". Clearly, the latter doesn't concern the situations in which one has to verify whether the map under consideration is continuous — this can be easily distinguished from the context.

Ordinals are the sets of the preceding ordinals. They are usually endowed with the order topology. The cardinal numbers are identified with the corresponding ordinal numbers. The unit segment $[0, 1]$ is often denoted by I, the countable power I^ω of I (the Hilbert cube), is denoted by Q.

We denote by βX the Stone-Čech compactification of a Tychonov space X. Given a map $f \colon X \to Y$, we denote by $\beta f \colon \beta X \to \beta Y$ the (unique) map that extends f.

For a locally compact space X, its Aleksandrov (one-point) compactification is denoted by αX.

1.1.1. Shchepin spectral theorem

In what follows, an *inverse system* $\mathcal{S} = \{X_\alpha, p^\alpha_\beta; \mathcal{A}\}$ satisfies the following conditions:

1) X_α are compact Hausdorff spaces;
2) p^α_β are surjective;
3) the partially ordered set $\mathcal{A}$ (by $\leq$) is directed, i.e., for every $\alpha, \beta \in \mathcal{A}$ there exists $\gamma \in \mathcal{A}$ with $\alpha \leq \gamma$, $\beta \leq \gamma$.

An inverse system $\mathcal{S} = \{X_\alpha, p^\alpha_\beta; \mathcal{A}\}$ is called *open* if all the maps p^α_β are open.

For every A, by $\mathcal{P}_\omega(A)$ we denote the family of all countable subsets of A ordered by inclusion.

An inverse system $\mathcal{S} = \{X_\alpha, p^\alpha_\beta; \mathcal{A}\}$ is called *continuous* if for every $\alpha \in \mathcal{A}$ we have $X_\alpha = \varprojlim\{X_{\alpha'}, p^{\alpha'}_{\beta'}; \alpha', \beta' < \alpha\}$.

By $\mathrm{w}(X)$ we denote the *weight* of a space X. An inverse system $\mathcal{S} = \{X_\alpha, p^\alpha_\beta; \mathcal{A}\}$ is called a *τ-system*, τ being a cardinal number, if the following holds:

1) the directed set $\mathcal{A}$ is τ-complete, i. e. every chain of cardinality $\leq \tau$ in $\mathcal{A}$ has the least upper bound;

2) $\mathcal{S}$ is continuous;

3) $\mathrm{w}(X_\alpha) \leq \tau$ for every $\alpha \in \mathcal{A}$.

If $\tau = \omega$, we use the terms *σ-complete* and *σ-system.*

A canonical example of σ-system is given by the following procedure. Suppose X is embedded in I^τ. For every $A \in \mathcal{P}_\omega(\tau)$ we denote by pr_A the projection map onto I^A, and for every $B \in \mathcal{P}_\omega(\tau)$, $A \subset B$, we denote by pr^B_A the projection map of I^B onto I^A. For every $A \in \mathcal{P}_\omega(\tau)$ let $X_A = \mathrm{pr}_A(X)$. Then $\mathcal{S} = \{X_A, \mathrm{pr}^B_A|X_B; \mathcal{P}_\omega(\tau)\}$ is a canonical σ-system with $\varprojlim \mathcal{S} = X$.

Let $\mathcal{A}$ be a directed partially ordered set. A subset $\mathcal{A}'$ of $\mathcal{A}$ is called *cofinal* if for every $\alpha \in \mathcal{A}$ there exists $\alpha' \in \mathcal{A}'$ such that $\alpha \leq \alpha'$. A subset $\mathcal{A}'$ of $\mathcal{A}$ is called *closed* if it contains all suprema of its chains.

Theorem 1.1.1. (Shchepin spectral theorem.) *Every map of the limit spaces of compact Hausdorff τ-systems defined over the same directed set is induced by a morphism of their cofinal closed subsystems. A homeomorphism of the limit spaces is induced by an isomorphism of their cofinal closed subsystems.*

Theorem 1.1.2. *Let $f\colon X \to Y$ be an open map of the limits of σ-systems $\{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ and $\{Y_\alpha, q_{\alpha\beta}; \mathcal{A}\}$. Then the set of all $\alpha \in \mathcal{A}$ with $fp^{-1}_\alpha = q^{-1}_\alpha f_\alpha$ is a cofinal and closed subset of $\mathcal{A}$.*

1.1.2. Dimension

The *order* $\operatorname{ord} \mathcal{A}$ of a family $\mathcal{A}$ of subsets of a space X is the maximal natural number n with the property: $\mathcal{A}$ contains a subfamily of cardinality $n+1$ with nonempty intersection. If such n doesn't exists, we say that $\operatorname{ord} \mathcal{A} = \infty$.

If X is a Tychonov space, then we say that $\dim X \leq n$ if for every finite cover $\mathcal{U}$ of X by functionally open sets there exists a finite cover $\mathcal{V}$ of X by functionally open sets such that $\operatorname{ord} \mathcal{V} \leq n+1$ and $\mathcal{V}$ is inscribed in $\mathcal{U}$.

Theorem 1.1.3. (Countable sum theorem.) *If X is a normal space and $X = \cup_{i=1}^{\infty} X_i$, where X_i are closed subsets of X with $\dim X_i \leq n$, then $\dim X \leq n$.*

Theorem 1.1.4. (Dowker theorem.) *Let X be a normal space, M a closed subset of X. If $\dim M \leq n$ and $\dim A \leq n$ for every closed set A, $A \cap M = \varnothing$, then $\dim X \leq n$.*

Definition 1.1.5. A map $f : X \to Y$ is called n-soft, $n = 0, 1, 2, \ldots$ if for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, hook] & X \arrow[d, "f"] \\ Z \arrow[r, "\varphi"] & Y, \end{tikzcd}$$

where A is a closed subset of a space Z, $\dim Z \leq n$ there exists a map $\Phi : Z \to X$ such that $\Phi | A = \psi$ and $f \Phi = \varphi$.

Removing all the dimensional restrictions onto the space Z from the above definition we obtain the notion of *soft* map.

Proposition 1.1.6. *Every 0-soft map of compact Hausdorff spaces is open. In the class of surjective maps of compact metrizable spaces the properties of openness and 0-softness are equivalent.*

1.1.3. Milutin maps

Let $f : X \to Y$ be an epimorphism of compact Hausdorff spaces. A linear regular (i. e. with norm 1) operator $u : C(X) \to C(Y)$ is called

an *averaging operator* for f if for every $\varphi \in C(Y)$ we have $u(\varphi \circ f) = \varphi$. If f admits an averaging operator, then f is called a *Milutin map.*

A compact Hausdorff space X is called a *Milutin space* if there exists a Milutin epimorphism $f\colon 2^\tau \to X$.

Theorem 1.1.7. (Milutin lemma.) *There exists a Milutin map of 2^ω onto the segment I.*

Proposition 1.1.8. *Suppose that epimormhisms $f_\alpha\colon X_\alpha \to Y_\alpha$ are Milutin maps for every $\alpha \in \mathcal{A}$. Then*

$$\prod\{f_\alpha \mid \alpha \in \mathcal{A}\}\colon \prod\{X_\alpha \mid \alpha \in \mathcal{A}\}\prod\{Y_\alpha \mid \alpha \in \mathcal{A}\}$$

is a Milutin map.

The following fact can be easily deduced from the Milutin lemma and Proposition 1.1.8.

Proposition 1.1.9. *For every compact Hausdorff space X there exists a Milutin epimorphism $f\colon Y \to X$, where Y is zero-dimensional.*

1.2. Category Theory

We denote the components of the natural transformations of functors in the operator form (e. g. $\varphi = (\varphi X)$) and usually do not use parentheses when denoting the action of the functors onto the objects. In this situation it is necessary to use the composition sign for morphisms.

If every morphism $\varphi X\colon F_1X \to F_2X$ of some natural transformation $\varphi = (\varphi X)\colon F_1 \to F_2$ is a monomorphism (respectively epimorphism) then F_1 is called a *subfunctor* (respectively a *quotient functor*) of F_2.

For a category $\mathcal{C}$ we denote by $|\mathcal{C}|$ the class of objects of $\mathcal{C}$. If $X, Y \in |\mathcal{C}|$, then $\mathcal{C}(X,Y)$ denotes the set of morphisms from X to Y in $\mathcal{C}$.

1.2.1. Monads

If T is an endofunctor in a category $\mathcal{C}$ and $\eta\colon 1_\mathcal{C} \to T$ and $\mu\colon T^2 \equiv TT \to T$ are natural transformations, then $\mathbb{T} = (T, \eta, \mu)$ is called a *monad* if

and only if the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, "1_T"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$$

Then η is called the *unity* and μ the *multiplication* of $\mathbb{T}$. The functor T is often refferred as the *functorial part* of $\mathbb{T}$.

If $\mathbb{T} = (T, \eta, \mu)$, $\mathbb{T}' = (T', \eta', \mu')$ are monads in a category C, then a natural transformation $\varphi: T \to T'$ is called a *morphism* of a monad $\mathbb{T}$ into a monad $\mathbb{T}'$, if the diagrams

$$\begin{tikzcd} 1_C \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & T \arrow[d, "\varphi"] \\ & T', \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "T\varphi\circ\varphi T"] \arrow[d, "\mu"'] & T'^2 \arrow[d, "\mu'"] \\ T \arrow[r, "\varphi"'] & T' \end{tikzcd}$$

are commutative.

If all the components φX of a morphism $\varphi: \mathbb{T} \to \mathbb{T}'$ are monomorphisms, we say that $\mathbb{T}$ is a *submonad* of $\mathbb{T}'$.

Definition 1.2.1. A monad (T, η, μ) on the category C is called *projective* if there exists a natural tranformation. (*projection*) $\pi: T \to 1_C$ such that $\pi \circ \eta = 1$, $\pi \circ \pi T = \pi \circ \mu$.

Note that a monad $\mathbb{T} = (T, \eta, \mu)$ in C is projective if and only if there exists a morphism of $\mathbb{T}$ into the identity monad $(1_C, 1_{1_C}, 1_{1_C})$.

Let C, $\mathcal{D}$ be categories and $L: C \to \mathcal{D}$, $R: \mathcal{D} \to C$ functors. We say that the functor L is *left adjoit* to R and R is *right adjoit* to L if for every $X \in |C|$, $Y \in |\mathcal{D}|$ there is a bijection $\Phi(X,Y): C(X, RY) \to \mathcal{D}(LX, Y)$ which is natural in the sense that for every $f: X \to X'$ in C and $g: Y \to Y'$ in $\mathcal{D}$ the diagram

$$\begin{tikzcd} C(X, RY) \arrow[r, "\Phi(X,Y)"] & \mathcal{D}(LX, Y) \\ C(X', RY') \arrow[u, "Rg\circ(-)\circ f"] \arrow[r, "\Phi(X',Y')"] & \mathcal{D}(LX', Y') \arrow[u, "Lf\circ(-)\circ g"'] \end{tikzcd}$$

is commutative. The morphisms $\Phi(X, LX)^{-1}(1_{LX}) \in \mathcal{C}(1_X, RLX)$, $X \in |\mathcal{C}|$, form the natural transformation $\eta = (\eta X)\colon 1_{\mathcal{C}} \to RL$ which is called the *unit* of the adjunction of L to R. Similarly, one can define the natural transformation $\varepsilon\colon LR \to 1_{\mathcal{D}}$ (the *counit* of the adjunction).

Theorem 1.2.2. *Suppose a functor $U\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint to a functor $F\colon \mathcal{D} \to \mathcal{C}$ and $\eta\colon 1_{\mathcal{D}} \to UF$, $\varepsilon\colon FU \to 1_{\mathcal{C}}$ are the unit and counit of this adjunction. Then the triple $\mathbb{T} = (UF, \eta, U\varepsilon F$ is a monad on $\mathcal{D}$.*

Suppose $C\colon \mathcal{C} \to \mathcal{C}$ is a contravariant functor such that there exists a natural transformation $\eta\colon 1 \to C^2$ satisfying the property: $C\eta \circ \eta C = 1_C$. Put $T = C^2$ and define the natural transformation $\mu\colon T^2 = C^4 \to C^2 = T$ by the formula: $\mu = C\eta C$.

Proposition 1.2.3. *$\mathbb{T} = (T, \eta, \mu)$ is a monad on the category $\mathcal{C}$.*

Proof. We have

$$\begin{aligned}\mu \circ T\eta &= C\eta C \circ C^2\eta = C(C\eta \circ \eta C) = 1_{C^2} = 1_T,\\ \mu \circ \eta T &= C\eta C \circ \eta C^2 = 1_{C^2} = 1_T.\end{aligned}$$

and

$$\begin{aligned}\mu \circ T\mu &= C\eta C \circ C^3\eta C = C(C^2\eta C \circ \eta C)\\ &= C(\eta C^3 \circ \eta C) = C\eta C \circ C\eta C^3 = \mu \circ \mu T.\end{aligned}$$

□

Let $\mathbb{T}, \eta, \mu)$ be a monad on a category $\mathcal{C}$. Let $S\colon \mathcal{C} \to \mathcal{D}$ be a functor for which there exists a natural transformation $\lambda\colon ST \to S$ making the diagrams

$$\begin{tikzcd} ST^2 \arrow[r, "S\mu"] \arrow[d, "\lambda T"'] & ST \arrow[d, "\lambda"] \\ ST \arrow[r, "\lambda"] & S, \end{tikzcd} \qquad \begin{tikzcd} ST \arrow[r, "\lambda"] & S \\ S \arrow[u, "S\eta"] \arrow[ur, "1"'] & \end{tikzcd}$$

commutative. Then S is called a *right* $\mathbb{T}$*-functor*.

1.2.2. Eilenberg-Moore category of a monad

For an arbitrary monad $\mathbb{T} = (T, \eta, \mu)$ in $\mathcal{C}$ a pair (X, f), where $f: TX \to X$ is a morphism in $\mathcal{C}$, is called a $\mathbb{T}$-*algebra* iff the following commute:

$$\begin{tikzcd} X \arrow[r, "\eta X"] \arrow[dr, "1_X"'] & TX \arrow[d, "f"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu X"] \arrow[d, "Tf"'] & TX \arrow[d, "f"] \\ TX \arrow[r, "f"] & X. \end{tikzcd}$$

The morphism $f: TX \to X$ is then refferred as the *structure morphism* of the $\mathbb{T}$-algebra (X, f).

Evidently, the couple $(TX, \mu X)$ is a $\mathbb{T}$-algebra for every X. This algebra is said to be a *free* $\mathbb{T}$-*algebra*, determined by the object X. An arrow $\varphi: X \to Y$ is called *a morphism of algebras* $(X, f) \to (Y, g)$ if and only if the diagram

$$\begin{tikzcd} TX \arrow[r, "T\varphi"] \arrow[d, "f"'] & TY \arrow[d, "g"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

commutes. $\mathbb{T}$-algebras and maps of algebras form the *Eilenberg-Moore category* $\mathcal{C}^{\mathbb{T}}$.

Note that for every morphism $f \in \mathcal{C}(X, Y)$ the morphism Tf is a morphism of the free $\mathbb{T}$-algebras $(TX, \mu X)$ and $(TY, \mu Y)$.

Proposition 1.2.4. *Suppose $\mathbb{T}$ is a monad in a category $\mathcal{C}$ with arbitrary products. Then the category $\mathcal{C}^{\mathbb{T}}$ is also a category with arbitrary products.*

Define the functors $F^{\mathbb{T}}: \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$, $U^{\mathbb{T}}: \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ by the data

$$\begin{aligned} F^{\mathbb{T}}X =& (TX, \mu X), \quad F^{\mathbb{T}}f = Tf, \; f \in \mathcal{C}(X, Y) \\ U^{\mathbb{T}}(X, \xi) =& X, \quad U^{\mathbb{T}}f = Tf, \; f \in \mathcal{C}^{\mathbb{T}}((X, \xi), (Y, \varsigma)). \end{aligned}$$

Theorem 1.2.5. *The functor $U^{\mathbb{T}}$ is left adjoint to the functor $F^{\mathbb{T}}$. The monad that corresponds to this adjunction is $\mathbb{T}$.*

A functor $\overline{F}: \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ is called a *lifting* of a functor $F: \mathcal{C} \to \mathcal{C}$ if $FU = U\bar{F}$, where $U = U^{\mathbb{T}}: \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is the forgetful functor.

Theorem 1.2.6. *There exists a bijective correspondence between the liftings of a functor F to the category $C^{\mathbb{T}}$ and the natural transformations $\delta\colon TF \to FT$ such that*
1) $\delta \circ \eta F = F\eta$;
2) $F\mu \circ \delta T \circ T\delta = \delta \circ \mu F$.

Sketch of the proof. Given a natural transformation $\delta\colon TF \to FT$ satisfying conditions 1) and 2), define the lifting $\overline{F}$ using the formula $\overline{F}(X,\xi) = (FX, F\xi \circ \delta X)$.

Conversely, given a lifting $\widetilde{F}$ of F, let $(FTX, \tilde{\mu}X) = \widetilde{F}(TX, \mu X)$ and define the natural transformation $\tilde{\delta}\colon TF \to FT$ by $\tilde{\delta} = \tilde{\mu} \circ TF\eta X$. □

1.2.3. Kleisli categories

The *Kleisli category of* $\mathbb{T}$ is the category $C_{\mathbb{T}}$ defined as follows: $|C_{\mathbb{T}}| = |C|$, $C_{\mathbb{T}}(X,Y) = C(X,TY)$, and the composition $g * f$ of morphisms $f \in C_{\mathbb{T}}(X,Y)$, $g \in C_{\mathbb{T}}(Y,Z)$ is given by $g * f = \mu Z \circ Tg \circ f$.

Note that the category $C_{\mathbb{T}}$ can be embedded into $C^{\mathbb{T}}$ as a full subcategory by means of the functor Φ:

$$\Phi X = (TX, \mu X), \quad \Phi f = \mu Y \circ Tf, \; f \in C_{\mathbb{T}}(X,Y).$$

(Thus, the image of Φ is the subcatefory of free $\mathbb{T}$-algebras.)

Define the functor $F_{\mathbb{T}}\colon C \to C_{\mathbb{T}}$ by $F_{\mathbb{T}}X = X$, $X \in |C|$ and $F_{\mathbb{T}}f = \eta Y \circ f$ for $f \in C(X,Y)$, and the functor $U_{\mathbb{T}}\colon C_{\mathbb{T}} \to C$ by $U_{\mathbb{T}}X = TX$, $U_{\mathbb{T}}f = \mu Y \circ Tf$, $f \in C_{\mathbb{T}}(X,Y)$.

Theorem 1.2.7. *The functor $U_{\mathbb{T}}$ is left adjoint to the functor $F_{\mathbb{T}}$. The monad that corresponds to this adjunction is $\mathbb{T}$.*

A functor $\overline{F}\colon C_{\mathbb{T}} \to C_{\mathbb{T}}$ called an *extension of the functor $F\colon C \to C$ on the Kleisli category $C_{\mathbb{T}}$* if $IF = \overline{F}I$.

Theorem 1.2.8. *There exists a bijective correspondence between extensions of functor F onto the Kleisli category $C_{\mathbb{T}}$ of monad $\mathbb{T}$ and natural transformations $\xi\colon FT \to TF$ satisfying*
1) $\xi \circ F\eta = \eta F$;
2) $\mu F \circ T\xi \circ \xi T = \xi \circ F\mu$.

Suppose that $t\colon F \to F'$ is a natural transformation of endofunctors in a category C, $\bar{F}$, $\bar{F}'$ are extensions of F and F' respectively onto

the category $\mathcal{C}_{\mathbb{T}}$. Is t a natural transformation of the extended functors? The following statement gives an affirmative answer to this question.

Definition 1.2.9. Suppose that $\xi\colon FT \to TF$, $\xi'\colon F'T \to TF'$ are the natural transformations corresponding to extensions $\bar{F}$, $\bar{F}'$ of functors F and F' respectively onto the category $\mathcal{C}_{\mathbb{T}}$. A natural transformation $t\colon F \to F'$ is called $\mathbb{T}$-*coordinated* if $Tt \circ \xi = \xi' \circ tT$.

Proposition 1.2.10. *A natural transformation $t\colon F \to F'$ is also a natural transformation of extended functors if and only if t is $\mathbb{T}$-coordinated.*

Proof. Necessity. Suppose that $f\colon X \to TY$ is a morphism in $\mathcal{C}$. Considering f as a morphism from X to Y in $\mathcal{C}_{\mathbb{T}}$, we obtain

$$\begin{aligned}
\bar{F}'f * tX =&\mu F'Y \circ T\xi'Y \circ TF'f \circ \eta F'X \circ tX\\
=&\mu F'Y \circ T\xi'Y \circ \eta F'TY \circ F'f \circ tX\\
=&\mu F'Y \circ T\xi'Y \circ \eta F'TY \circ TtY \circ Ff\\
=&\mu F'Y \circ T^2tY \circ T\xi Y \circ \eta FTY \circ Ff\\
=&\mu F'Y \circ T^2tY \circ \eta TFY \circ \xi Y \circ Ff\\
=&\mu F'Y \circ \eta TF'Y \circ TtY \circ \xi Y \circ Ff\\
=&\mu F'Y \circ T\eta F'Y \circ TtY \circ \xi Y \circ Ff = tY * \bar{F}f,
\end{aligned}$$

i. e. $t\colon \bar{F} \to \bar{F}'$ is a natural transformation.

Sufficiency. Taking in the above arguments $f = 1_{TY}\colon TY \to TY$, we obtain

$$\begin{aligned}
\xi Y \circ TtY &= \mu F'Y \circ T\eta F'Y \circ TtY \circ \xi Y \circ 1_{FTY}\\
&= \mu F'Y \circ T\xi'Y \circ \eta F'TY \circ tTY \circ 1_{FTY}\\
&= \mu F'Y \circ \eta TF'Y \circ \xi Y \circ tTY = \xi Y \circ tTY,
\end{aligned}$$

i.e. the natural transformation t is $\mathbb{T}$-coordinated. □

Proposition 1.2.11. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. The natural transformation $\xi = \eta T \circ \mu\colon T^2 \to T^2$ is associated with an extension of the functor T onto $\mathcal{C}_{\mathbb{T}}$.*

Proposition 1.2.12. *Suppose that a natural transformation $\xi\colon T^2 \to T^2$ associated with some extension of T into $\mathcal{C}_{\mathbb{T}}$ satisfies the condition*

$$\xi \circ \mu T = T\mu \circ \xi T \circ T\xi. \tag{1.1}$$

Then for every natural k the natural transformation

$$\tilde{\xi}_k = \xi T^{k-1} \circ T\xi T^{k-2} \circ \cdots \circ T^{k-2}\xi T \circ T^{k-1}\xi$$

is associated with an extension of the functor T^k onto the category $C_{\mathbb{T}}$.

Proof. This can be checked by immediate calculations. □

Proposition 1.2.13. *The natural transformation $\mu T^{k-2}: T^k \to T^{k-1}$, $k \geq 2$, is $\mathbb{T}$-coordinated with respect to the extensions of the functors T^k, T^{k-1} onto the category $C_{\mathbb{T}}$ associated with the natural transformations $\tilde{\xi}_k$, $\tilde{\xi}_{k-1}$ respectively.*

Proof. Using condition 1.1 we obtain

$$\begin{aligned}
&\tilde{\xi}_{k-1} \circ \mu T^{k-2}\\
&= \xi T^{k-2} \circ T\xi T^{k-3} \circ \cdots \circ T^{k-3}\xi T \circ T^{k-2}\xi \circ \mu T^{k-1}\\
&= \xi T^{k-2} \circ T\xi T^{k-3} \circ \cdots \circ T^{k-3}\xi T \circ \mu T^{k-1} \circ T^{k-1}\xi = \ldots\\
&= \xi T^{k-2} \circ \cdots \circ \mu T^{k-1} T^2 \circ T^{k-3} \circ \cdots \circ T^{k-2}\xi T \circ T^{k-1}\xi\\
&= T\mu T^{k-2} \circ \xi T^{k-1} \circ T\xi T^{k-2} \circ T^2\xi T^{k-3} \circ \cdots \circ T^{k-2}\xi T \circ T^{k-1}\xi\\
&= \tilde{\xi}_{k-1} \circ \mu T^{k-1}.
\end{aligned}$$

□

A category C is called *complete,* if it contains limits of all diagrams. Suppose that C is a complete category and $\mathbb{T} = (T, \eta, \mu)$ is a monad in C. Using transfinite induction we can define, for every ordinal $\alpha \geq 1$ the *α-th iteration* of the functor T and the natural transformations

$$\mu_{\alpha,\beta}: T^\alpha \to T^\beta \quad \alpha \geq \beta.$$

Let

$$T^1 = T,\ T^{\alpha+1} = TT^\alpha,\ \mu_{\alpha,\alpha} = 1_{T^\alpha},\ \mu_{\alpha+1,\alpha} = \mu_{T^\alpha}.$$

Suppose that α is a limit ordinal and for every α', β', $\alpha > \alpha' \geq \beta'$ we have already defined $T^{\alpha'}$ and $\mu_{\alpha',\beta'}$. Then we put

$$(T^\alpha, \mu_{\alpha,\alpha'}) = \lim\{T^{\alpha'}, \mu_{\alpha',\beta'}; \alpha', \beta' < \alpha\}.$$

Denote by $\mathcal{S}^{\alpha}_{X}$ the inverse system $\{T^{\alpha'}X, \mu_{\alpha'\beta'}X; \alpha'\beta' < \alpha\}$.

Given a natural transformation $\xi\colon T^2 \to T^2$, we define, for every ordinal $\alpha \geq 1$ a natural transformation

$$\tilde{\xi}_{\alpha}\colon TT^{\alpha} \to T^{\alpha}T$$

so that the following holds:

1) $\tilde{\xi}_1 = \xi$;

2) if $\alpha = (\alpha - 1) + 1$, then

$$\tilde{\xi}_{\alpha} = \tilde{\xi}_{\alpha-1}T \circ T^{\alpha-1}\xi;$$

3) if $\beta' \leq \beta$, then the diagram

$$\begin{tikzcd}
T^{\beta}T \arrow[r, "\tilde{\xi}_{\beta}"] \arrow[d, "\mu_{\beta\beta'}T"'] & TT^{\beta} \arrow[d, "T\mu_{\beta\beta'}"] \\
T^{\beta'}T \arrow[r, "\tilde{\xi}_{\beta'}"'] & TT^{\beta'}
\end{tikzcd}$$

is commutative (i. e. for every X the collection $(\tilde{\xi}_{\beta})_{\beta<\alpha}$ is a morphism of an inverse system $\mathcal{S}^{\alpha}_{TX}$ into the inverse system $T(\mathcal{S}^{\alpha}_{X})$ and $\tilde{\xi}_{\alpha} = \varprojlim\{\tilde{\xi}_{\beta}X \mid \beta < \alpha\}$.

Proposition 1.2.14. *The natural transformation $\tilde{\xi}_{\alpha}$ is associated with an extension of the functor T^{α} to the category $\mathcal{C}_{\mathbb{T}}$.*

Proof. Immediate. □

Remark 1.2.15. The natural transformation $\xi = \eta T \circ \mu\colon T^2 \to T^2$ satisfies condition 1.1.

1.2.4. Extensions of contravariant functors onto the Kleisli categories

The following result is a counterpart of a result of Vinárek.

Proposition 1.2.16. *There exists a bijective correspondence between extensions of a contravariant functor $F: C \to C$ onto the category $C_{\mathbb{T}}$ and the natural transformations $\xi: F \to TFT$ satisfying the conditions:*
1) $TF\eta \circ \xi = \eta F$;
2) $TF\mu \circ \xi = \mu FT^2 \circ T\xi T \circ \xi$.

Proof. Suppose there is a natural transformations $\xi: F \to TFT$ such that conditions (i) and (ii) are satisfied. For every $X \in |C|$ put $\overline{F}X = FX$ and for every $f \in C_{\mathbb{T}}(X, Y)$ put $\overline{F}f = TFf \circ \xi Y$.

If $f = Ig$ for $g \in C(X, Y)$, we obtain

$$\begin{aligned}\overline{F}f =&\overline{F}(\eta Y \circ g) = TF(\eta Y \circ g) \circ \xi Y = TFg \circ TF\eta Y \circ \xi Y \\ =&TFg \circ \eta FY = \eta FX \circ Fg = IFg.\end{aligned}$$

For $f \in C_{\mathbb{T}}(X, Y)$ and $g \in C_{\mathbb{T}}(Y, Z)$ we obtain

$$\begin{aligned}\overline{F}(g * f) =&TF(g * f) \circ \xi Z = TF(\mu Z \circ Tg \circ f) \circ \xi Z \\ =&TFf \circ TFTg \circ TF\mu Z \circ \xi Z \\ =&TFf \circ TFTg \circ \mu FT^2 Z \circ T\xi TZ \circ \xi Z \\ =&TFf \circ \mu FTY \circ T^2FTg \circ T\xi TZ \circ \xi Z \\ =&TFf \circ \mu FTY \circ T\xi Y \circ TFg \circ \xi Z \\ =&\mu FX \circ T^2Ff \circ T\xi Y \circ TFg \circ \xi Z \\ =&\mu FX \circ T(TFf \circ \xi Y) \circ TFg \circ \xi Z = \overline{F}f * \overline{F}g.\end{aligned}$$

Summing up we see that $\overline{F}$ is an extending F contravariant endofunctor in $C_{\mathbb{T}}$.

On the other hand, suppose $\overline{F}: C_{\mathbb{T}} \to C_{\mathbb{T}}$ is an extension of F onto $C_{\mathbb{T}}$. Since $1_{TX} \in C_{\mathbb{T}}(TX, X)$, we see that $\overline{F}1_{TX} \in C_{\mathbb{T}}(FX, FTX) = C(FX, TFTX)$. Put $\xi X = \overline{F}1_{TX}$.

Show that $\xi = (\xi X)$ is a natural transformation from F to TFT. Given $f \in C(X, Y)$ we obtain

$$\begin{aligned}\overline{F}(If * 1_{TX}) =&\overline{F}1_{TX} * \overline{F}If = \overline{F}1_{TX} * IFf = \mu FTX \circ T\xi X \circ \eta FX \circ Ff \\ =&\mu FTX \circ \eta TFTX \circ \xi X \circ Ff = \xi X \circ Ff\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\overline{F}(If * 1_{TX}) =&\overline{F}(\mu Y \circ T(\eta Y \circ f) \circ 1_{TX}) = \overline{F}(\mu Y \circ T1_{TY} \circ \eta TY \circ Tf) \\ =&\overline{F}(1_{TY} * ITf) = \overline{F}ITf * \overline{F}1_{TY} = IFTf * \xi Y \\ =&(\eta FTX \circ FTf) * \xi Y = \mu FTX \circ T\eta FTX \circ TFTf \circ \xi Y \\ =&TFTf \circ \xi Y.\end{aligned}$$

Thus, $TFTf \circ \xi Y = \xi X \circ Ff$.

Show that (i) holds. We have

$$\begin{aligned}\eta FX = IF1_X =& \overline{F}I1_X = \overline{F}(I1_X * I1_X) = \overline{F}(\eta X * \eta X)\\ =& \overline{F}(\mu X \circ T\eta X \circ \eta X) = \overline{F}(\mu X \circ T(1_{TX}) \circ \eta TX \circ \eta X)\\ =& \overline{F}(1_{TX} * I\eta X) = \overline{F}I\eta X * \overline{F}1_{TX} = IF\eta X * \xi X\\ =& \mu FX \circ T\eta FX \circ TF\eta X \circ \xi X = TF\eta X \circ \xi X.\end{aligned}$$

Finally, we have to check (ii). We have

$$\begin{aligned}\mu FT^2X \circ T\xi TX \circ \xi X =& \xi TX * \xi X = \overline{F}(1_{T^2X}) * \overline{F}(1_{TX}) = \overline{F}(1_{TX} * 1_{T^2X})\\ =& \overline{F}(\mu X \circ T1_{TX} \circ 1_{T^2X}) = \overline{F}\mu X\\ =& \overline{F}(1_{TX} * I\mu X) = \overline{F}I\mu X * \overline{F}1_{TX} = IF\mu X * \xi X\\ =& \mu FT^2X \circ T\eta FT^2X \circ TF\mu X \circ \xi X = TF\mu X \circ \xi X.\end{aligned}$$

Show that the above correspondence is a bijection. Given a natural transformation $\xi = (\xi X)$ satisfying (i) and (ii) consider the extension $\overline{F}$ defined by $\overline{F}f = TFf \circ \xi Y$, where $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$. Then $\overline{F}$ determines the natural transformation $\hat{\xi} = (\hat{\xi}X)$, $\hat{\xi}X = \overline{F}1_{TX}$ and we have $\hat{\xi}X = \overline{F}1_{TX} = TF1_{TX} \circ \xi X = \xi X$.

Conversely, given an extension $\overline{F}$ of F onto the category $\mathcal{C}_{\mathbb{T}}$, consider the natural transformation $\xi = (\xi X)$ defined by $\xi X = \overline{F}1_{TX}$, $X \in |\mathcal{C}|$. The natural transformation ξ determines the extension $\hat{F}$ of F onto $\mathcal{C}_{\mathbb{T}}$ by the formula $\hat{F}f = TFf \circ \xi Y$, $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$. We have

$$\begin{aligned}\hat{F}f =& TFf \circ \xi Y = TFf \circ \overline{F}1_{TY} = \mu FX \circ T\eta FX \circ TFf \circ \overline{F}1_{TY}\\ =& \mu FX \circ T(\eta FX \circ Ff) \circ \overline{F}1_{TY} = IFf * \overline{F}1_{TY}\\ =& \overline{F}If * \overline{F}1_{TY} = \overline{F}(1_{TY} * If) = \overline{F}(1_{TY} * (\eta TY \circ f))\\ =& \overline{F}(\mu Y \circ T1_{TY} \circ \eta TY \circ f) = \overline{F}f.\end{aligned}$$

□

1.3. Notes and comments to Chapter 1.2

The Shchepin spectral theorem is proved by E. V. Shchepin [1976]. A detailed exposition of the material concerning the Shchepin theorem and its applications can be found in V. V. Fedorchuk, V. V. Filippov [1988]. See A. Pełczyński [1968] for the notion of Milutin map and the proof of the Milutin lemma.

The proofs of the results of dimension theory can be found, e. g. in J. van Mill [1989], R. Engelking [1978]. The notion of (n-)soft map is introduced by E. V. Shchepin [1981].

Invention of monads is attributed to R. Godement [1958]. It was proved by S. Eilenberg and J. C. Moore [1964] and H. Kleisli [1965] that every monad is induced by a pair of adjoint functors.

Theorem 1.2.2 is proved by P. Huber [1961].

Theorem 1.2.8 is due to M. Arbib and E. Manes [1975] and J. Vinárek [1983]. For Theorem 1.2.6 see M. Zarichnyi [1991b].

Definition 1.2.1 of projective monad is due to J. Vinárek [1983].

Proposition 1.2.16 is proved by V. Levyts'ka [1998].

List of categories

Notation	Objects	Morphisms
Set	sets	maps
Top	topological spaces	continuous maps
Comp	compact Hausdorff spaces	continuous maps
$\mathbf{Comp}_0$	zero-dimensional compact Hausdorff spaces	continuous maps
Conv	convex compact spaces	affine continuous maps
$\mathbf{K}_n$	compact Hausdorff spaces of cardinality $\leq n$	continuous maps
Tych	Tychonov spaces	continuous maps
Cgrp	compact Hausdorff groups	continuous homomorphisms
CISG	compact Hausdorff inverse semigroups	continuous homomorphisms
Unif	uniform spaces	uniformly continuous maps
Metr	compact metric spaces	nonexpanding maps
NF	normal functors	natural transformations
$\mathbf{NF}_{\mathbf{Tych}}$	normal functors in **Tych**	natural transformations
WNF	weakly normal functors	natural transformations
ANF	almost normal functors	natural transformations

Chapter 2.

Normal functors

This chapter is devoted to systematic investigations of the class of normal functors and related classes of functors. We begin our exposition with some examples of functors (Section 2.1), then consider some elementary properties of functors (Section 2.2). The definition of normal functor appears in Section 2.3. Section 2.7 is devoted to Chigogidze's construction of extension of normal functors onto the category **Tych**. In Section 2.9 a characterization theorem for the power functors is proved. In Section 2.10 we detailly investigate the properties of openness and bicommutativity of functors; in the class of normal functors of finite degree these properties are characteristic ones for the G-symmetric power functors. The main concern of Section 2.11 is to introduce the notion of characteristic of normal functors; some applications will be given in Chapter 5.

2.1. Examples of functors in the category of compact Hausdorff spaces

2.1.1. Hyperspaces

Let X be a Tychonov space. The *hyperspace* $\exp X$ of X is the space of all nonempty compact subsets in X endowed with the so-called *Vietoris topology*. A base of this topology consists of the sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset U_1 \cup \dots \cup U_n \\ \text{and } A \cap U_i \neq \varnothing \text{ for every } i\},$$

where $U_1, \ldots, U_n$ run through the topology of X. Note that since

$$\langle U_1, \ldots, U_n \rangle = \langle U_1 \cup \cdots \cup U_n \rangle \cap \langle U_1, X \rangle \cap \cdots \cap \langle U_n, X \rangle,$$

the base sets of the form $\langle U \rangle$ and $\langle X, U \rangle$ form a subbase of the Vietoris topology.

Proposition 2.1.1. *The space* $\exp X$ *is compact Hausdorff provided so is* X.

Proof. We need the Alexander lemma: a Tychonov space is compact if and only if it posesses a subbase such that each open cover of this space by subbase elements contains a finite subcover. □

For a map $f\colon X \to Y$ of Tychonov spaces the map $\exp f$ is defined by the formula $\exp f(A) = f(A)$, $A \in \exp X$. Since $(\exp f)^{-1}(\langle U \rangle) = \langle f^{-1}(U) \rangle$, $(\exp f)^{-1}(\langle Y, U \rangle) = \langle X, f^{-1}(U) \rangle$, we see that the map $\exp f$ is continuous. We obtain the *hyperspace functor in* **Tych**. By Proposition 2.1.1, $\exp|$**Comp** is an endofunctor in **Comp**. We preserve the notation $\exp$ for $\exp|$**Comp**.

By $\exp^c X$ we denote the subspace of $\exp X$ consisting of all connected sets. It is easy to see that $\exp^c X$ is closed in $\exp X$. Since $\exp f(\exp X) \subset \exp Y$, we can define a subfunctor $\exp^c$ of $\exp$ (in **Tych** and **Comp**).

Proposition 2.1.2. *Let* $\mathcal{A} \in \exp^2 X$, *then* $\cup\mathcal{A} \in \exp X$.

Proof. Suppose $x \notin \cup\mathcal{A}$. For every $A \in \mathcal{A}$ there is a neighborhood U_A of x such that $A \in \langle X \setminus \bar{U}_A \rangle$. The open cover $\{\langle X \setminus \bar{U}_A \rangle \mid A \in \mathcal{A}$ contains a finite subcover $\{\langle X \setminus \bar{U}_1 \rangle, \ldots, \langle X \setminus \bar{U}_k \rangle\}$. Then

$$\mathcal{A} \subset \langle X \setminus \bar{U}_1 \rangle \cup \cdots \cup \langle X \setminus \bar{U}_k \rangle \subset \langle X \setminus (\bar{U}_1 \cap \cdots \cap \bar{U}_k) \rangle$$

and therefore the neighborhood $U_1 \cap \cdots \cap U_k$ of x does not intersect the set $\cup\mathcal{A}$. □

Thus, the *union map* $\cup = uX\colon \exp^2 X \to \exp X$ is well-defined.

Proposition 2.1.3. *The map* uX *is continuous.*

Proof. This follows from the equalities

$$(uX)^{-1}(\langle U \rangle) = \langle \langle U \rangle \rangle, \quad (uX)^{-1}(\langle X, U \rangle) = \langle \exp X, \langle X, U \rangle \rangle.$$

□

If $f\colon X \to Y$ is an onto map in **Comp**, then the inverse map $f^{-1}\colon Y \to \exp X$ is defined.

Proposition 2.1.4. *The map $f^{-1}\colon Y \to \exp X$ is continuous if and only if the map $f\colon X \to Y$ is open.*

Proof. If the map f is open then the set

$$(f^{-1})^{-1}(\langle X, U\rangle) = \{y \in Y \mid f^{-1}(y) \cap U \neq \varnothing\} = f(U) \tag{2.1}$$

is open for every open map $U \subset X$. Since f is a closed map, the set

$$(f^{-1})^{-1}(\langle U\rangle) = \{y \in Y \mid f^{-1}(y) \subset U\} = Y \setminus f^{-1}(f(X \setminus U)).$$

Conversely, continuity of the map f^{-1} and formula (2.1) imply that the set $f(U)$ is open for every open $U \subset X$. □

A map $f\colon X \to \exp Y$ is called *upper (lower) semicontinuous* if for every open subset U of Y the set $f_{\#}(U) = \{x \in X \mid f(x) \subset U\}$ (respectively $f^{\#}(U) = \{x \in X \mid f(x) \cap U \neq \varnothing\}$) is open in X. It is easy to see that any map $f\colon X \to \exp Y$ is continuous if it is both upper and lower semicontinuous.

Exercises

1. For every continuous pseudometric d on X define a function $d_{\mathrm{H}}\colon \exp X \times \exp X \to \mathbb{R}$ by the formula:

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

Prove that if d is a (pseudo)metric then so is d_{H} (the *Hausdorff metric*.

Let $\{d_\alpha \mid \alpha \in A\}$ be a family of continuous pseudometrics on $X \in |\mathbf{Comp}|$ generating its topology. Prove that $\{d_{\alpha\mathrm{H}} \mid \alpha \in A\}$ be a family of continuous pseudometrics on $\exp X$ generating its topology.

2. Let $\mathfrak{L}$ be a subset of $\exp^2 X$ consisting of families with nonempty intersection. Show that the map $\cap\colon \mathfrak{L} \to \exp X$ is not necessarily continuous.
3. Denote by $\exp^c_k X$ the subspace of $\exp X$ consisting of sets with at most k components, $k \in \mathbb{N}$. Prove that $\exp^c_k X$ is closed in $\exp X$ and that $\exp f(\exp^c_k X) \subset \exp^c_k Y$ for every map $f\colon X \to Y$.
4. Let $f\colon X \to Y$ be an onto map of compact Hausdorff spaces. Prove that the inverse map $f^{-1}\colon Y \to \exp X$ is upper semicontinuous.

2.1.2. Hyperspaces of convex compacta

We say that a compact Hausdorff space is *convex* if its topology and fixed convex structure are induced by some embedding into a locally convex Hausdorff linear topological space. Denote by **Conv** the category of all convex compacta and their affine continuous maps. Let $U\colon \mathbf{Conv} \to \mathbf{Comp}$ be a forgetful functor. For a convex compact Hausdorff space X consider the subspace $\mathrm{cc}\, X$ of $\exp X$ formed with all convex closed subsets of X. It is easy to verify that this subspace is closed in $\exp X$, thus, $\mathrm{cc}\, X \in \mathbf{Comp}$. Since for an affine continuous map $f\colon X \to Y$ of convex compacta $\exp f(\mathrm{cc}\, X) \subset \mathrm{cc}\, Y$, we can define a map $\mathrm{cc}\, f\colon \mathrm{cc}\, X \to \mathrm{cc}\, Y$ as the restriction of $\exp f$ on $\mathrm{cc}\, X$. We obtain the functor $\mathrm{cc}\colon \mathbf{Conv} \to \mathbf{Comp}$.

2.1.3. Inclusion hyperspaces, linked systems, and superextensions

Now, let X be a compact Hausdorff space. A closed subset $\mathcal{A}$ in $\exp X$ is called an *inclusion hyperspace* if the following condition holds: for every $A \in \mathcal{A}$ and every $B \in \exp X$ the inclusion $A \subset B$ implies $B \in \mathcal{A}$. Denote by GX the space of all inclusion hyperspaces with the induced from $\exp^2 X$ topology.

Let $U \subset X$. Put

$$\begin{aligned} U^+ &= \{\mathcal{A} \in GX \mid \text{ there exists } A \in \mathcal{A} \text{ such that } A \subset U\}, \\ U^- &= \{\mathcal{A} \in GX \mid A \cap U \neq \varnothing \text{ for every } A \in \mathcal{A}\}. \end{aligned}$$

Proposition 2.1.5. *The sets*

$$(U_1^+ \cap \cdots \cap U_m^+) \cap (V_1^- \cap \cdots \cap U_n^-),$$

where $U_1, \ldots, U_m, V_1, \ldots, U_n$ are open in X form a base of topology in GX.

Proof. The sets

$$\langle \langle W_{11}, \ldots, W_{1n_1}\rangle, \ldots, \langle W_{k1}, \ldots, W_{kn_k}\rangle\rangle,$$

where W_{ij} are open in X, form a base of the topology in $\exp^2 X$. Let $\mathcal{W} = \{W_{ij} \mid 1 \le i \le k,\ 1 \le j \le n_k\}$.

Suppose that

$$\mathcal{A} \in GX \cap \langle\langle W_{11}, \ldots, W_{1n_1}\rangle, \ldots, \langle W_{k1}, \ldots, W_{kn_k}\rangle\rangle.$$

A subfamily $\mathcal{V}$ of $\mathcal{W}$ is called *distinguished* if there is $A \in \mathcal{A}$ such that $\mathcal{V} = \{W \in \mathcal{W} \mid A \cap W \neq \varnothing\}$.

Then

$$\begin{aligned}\mathcal{A} \in & \cap \{(\cup \mathcal{V})^- \mid \mathcal{V} \text{ is distinguished}\} \\ & \cap (W_{11} \cup \cdots \cup W_{1n_1})^+ \cap \cdots \cap (W_{k1} \cup \cdots \cup W_{kn_k})^+ \\ & \subset \langle\langle W_{11}, \ldots, W_{1n_1}\rangle, \ldots, \langle W_{k1}, \ldots, W_{kn_k}\rangle\rangle.\end{aligned}$$

□

For a map $f\colon X \to Y$ we define the map $Gf\colon GX \to GY$ by

$$Gf(\mathcal{A}) = \{B \in \exp Y \mid B \supset f(A) \text{ for some } A \in \mathcal{A}\}, \quad \mathcal{A} \in GX.$$

Note that, since $(Gf)^{-1}(U^{\pm}) = (f^{-1}(U))^{\pm}$, by Proposition 2.1.5 the map Gf is continuous. It is easy to see that if a map f is not onto, then the map Gf does not coincide with the restriction of the map $\exp^2 f$ on GX.

Proposition 2.1.6. *The space GX is a closed subspace in* $\exp^2 X$.

Proof. Suppose $\mathcal{A} \in \exp^2 X \setminus GX$. Then there exists $A, B \in \exp X$ such that $A \subset B$, $A \in \mathcal{A}$ and $B \notin \mathcal{A}$. Since B is an element of the closure (in $\exp X$) of the set $\{C \in \exp X \mid B \subset \operatorname{Int} C\}$, there exists $C' \in \exp X$ such that $C' \notin \mathcal{A}$ and $B \subset \operatorname{Int} C'$. Let

$$U = \big\langle \exp X \setminus \{C'\}, \langle \operatorname{Int} C'\rangle\big\rangle,$$

then U is a neighborhood of $\mathcal{A}$ in $\exp^2 X$ such that $U \cap GX = \varnothing$. □

Corollary 2.1.7. *The space GX is compact for every compact Hausdorff space X.*

Let $\mathcal{A}, \mathcal{B} \in \exp^2 X$. The subsets in GX of the form

$$H(\mathcal{A}, \mathcal{B}) = \bigcap\{A^+ \mid A \in \mathcal{A}\} \cap \big(\bigcap\{B^- \mid B \in \mathcal{B}\}\big)$$

will be called *convex*. The subset of all nonempty convex subsets in GX will be denoted by KGX.

For every nonempty subset $\mathfrak{A} \subset GX$ we have $\cap\mathfrak{A}, \cup\mathfrak{A} \in GX$. It easily follows from Proposition 2.1.3 that the map $\mathfrak{A} \mapsto \cup\mathfrak{A}$ is continuous as a map from $\exp GX$ to GX.

Proposition 2.1.8. *The map* $\cap\colon \exp GX \to GX$, $\mathfrak{A} \mapsto \cap\mathfrak{A}$, *is continuous.*

Proof. This follows from the equalities

$$\cap^{-1}(U^+) = \langle U^+\rangle,\ \cap^{-1}(U^-) = \langle GX, U^-\rangle,$$

where $U \subset X$. □

For a map $f\colon X \to Y$ maps $N_k f\colon N_k X \to N_k Y$ and $\lambda f\colon \lambda X \to \lambda Y$ are restrictions of Gf onto the respective subspaces.

Proposition 2.1.9. *The maps* $f, g\colon \exp X \to \exp GX$, $f(A) = A^+$, $g(A) = A^-$, *are continuous.*

Proof. If $U_1, \dots, U_m, V_1, \dots, V_n$ are open subsets in X, then the set

$$f^{-1}(\langle U_1^+ \cap \dots \cap U_m^+ \cap V_1^- \cap \dots \cap V_n^-\rangle)$$

equals $\cap_{i=1}^m \langle U_i\rangle$, whenever $V_1 = \dots = V_n = X$, and equals $\varnothing$, otherwise. Besides,

$$f^{-1}(\langle GX, U_1^+ \cap \dots \cap U_m^+ \cap V_1^- \cap \dots \cap V_n^-\rangle) = \langle X, V_1, \dots, V_n\rangle \cap \cap_{i=1}^m \langle U_i\rangle.$$

This implies continuity of f. The case of g is left to the reader. □

Define a map $rX \exp^2 X \to GX$ by the formula

$$rX(\mathcal{A}) = \{B \in \exp X \mid B \supset A \text{ for some } A \in \mathcal{A}\}.$$

Lemma 2.1.10. *Let* $\mathcal{M} \in \exp GX$. *Then for every* $C \in \exp X$ *the following conditions are equivalent: a)* $\bigcap \mathcal{M} \in C^-$; *b)* $\mathcal{M} \cap C^- \neq \varnothing$.

Proof. a)⇒b). Let V be a neighborhood of the set C in X. Then $(X \setminus V) \notin \bigcap \mathcal{M}$, and hence, there exists $A \in \mathcal{M}$ such that $A \in \overline{V}$, i.e., $\mathcal{M} \cap \overline{V}^- \neq \varnothing$. Since C is contained in the closure (in $\exp X$) of the family

$$\{\overline{V} \mid V \text{ is a neighborhood of } C \text{ in } X\}$$

and the map $(-)^-\colon \exp X \to \exp GX$ is continuous (see Proposition 2.1.9), it follows from the closedness of the set $\mathcal{M}$ in GX that $\mathcal{M} \cap C^- \neq \varnothing$.

b)⇒a). If $A \in \mathcal{M} \cap C^-$, then $\bigcap \mathcal{M} \subset A$, and hence, $\bigcap \mathcal{M} \in C^-$. □

Proposition 2.1.11. *The map rX is a continuous retraction of $\exp^2 X$ onto GX.*

Proof. Obviously, rX is a retraction. To prove the continuity, note that

$$(rX)^{-1}(U^+) = \langle\langle U\rangle, \exp X\rangle,\ (rX)^{-1}(U^-) = \langle\langle U, X\rangle\rangle.$$

□

It can be directly verified that $r = (rX)$ is a natural transformation from the functor $\exp^2$ to G.

By $N_k X$, $k \geq 2$, we denote the subspace of GX consisting of all k-linked systems of closed subsets of X (the system is called *k-linked* if the intersection of every its k-element subsystem is nonempty). The elements of $N_2 X$ are said to be *full linked systems* (sometimes, *complete chained systems*). The notation NX is also used for $N_2 X$.

A full linked system is a *maximal linked system* if it is maximal with respect to inclusion. A subspace of all maximal linked systems in $N_2 X$ is called a *superextension* of X (written λX).

Proposition 2.1.12. *Let U be either open or closed subset in X. Then $U^+ \cap \lambda X = U^- \cap \lambda X$.*

Proof. Suppose first that U is closed in X and $\mathcal{M} \in \lambda X \cap U^+$. Then there exists $M \in \mathcal{M}$, $M \subset U$. Since $\mathcal{M}$ is linked, we see that $N \cap U \supset N \cap M \neq \varnothing$ for every $N \in \mathcal{M}$, i. e. $\mathcal{M} \in U^-$. On the other hand, if $\mathcal{M} \in \lambda X \cap U^-$, then, by maximality of $\mathcal{M}$, we have $U \in \mathcal{M}$, i. e. $\mathcal{M} \in \lambda X \cap U^+$.

For an open subset U of X we have an obvious inclusion $U^+ \cap \lambda X \subset U^- \cap \lambda X$. To prove the reverse inclusion, consider $\mathcal{M} \in \lambda X \cap U^-$ and show that there exists $B \in \exp X$, $B \subset U$, such that $\mathcal{M} \in B^-$. Assuming the contrary show that for every open subset $U' \subset U$ such that $\bar{U}' \subset U$ we have $X \setminus U' \in \mathcal{M}$. Indeed, there exists an open subset V such that $\bar{U}' \subset V \subset \bar{V} \subset U$ and then there exists an element $M \in \mathcal{M}$ such that $\bar{V} \cap M = \varnothing$. However, $X \setminus U' \supset M$ and therefore $X \setminus U' \in \mathcal{M}$. We have only to remark that $X \setminus U'$ belongs to the closure (in $\exp X$) of the family

$$\{X \setminus U' \mid U' \subset U \text{ is open and } \bar{U}' \subset U\}$$

and thus $X \setminus U \in \mathcal{M}$, a contradiction.

Finally, for some $B \in \exp X$, $B \subset U$, we have

$$\mathcal{M} \in B^- \cap \lambda X = B^+ \cap \lambda X \subset U^+ \cap \lambda X.$$

□

Thus, the sets A^+, where A is open (closed) in X, form open (closed) subbase in λX.

Proposition 2.1.13. *The map $f = (-)^+ \colon \exp X \to \exp \lambda X$, $A \mapsto A^+$, is continuous.*

Proof. Let $U_1, \ldots, U_n$ be open subsets of X. The assertion follows from the equalities

$$f^{-1}(\langle U_1^+ \cap \cdots \cap U_n^+ \rangle) = \langle U_1 \cap \cdots \cap U_n \rangle,$$
$$f^{-1}(\langle U_1^+ \cap \cdots \cap U_n^+, \lambda X \rangle) = \langle U_1 \cap \cdots \cap U_n, X \rangle.$$

□

For every $\mathcal{A} \in GX$ let

$$\perp X(\mathcal{A}) = \{B \in \exp X \mid B \cap A \neq \varnothing \text{ for every } A \in \mathcal{A}\}.$$

It can be easily verified that $\perp X(\mathcal{A}) \in GX$, so we can define the *transversality map* $\perp X \colon GX \to GX$. Since $(\perp X)^{-1}(A^\pm) = A^\mp$, the map $\perp X$ is continuous.

Lemma 2.1.14. *For any $\mathcal{A} \in GX$,*

$$\{\mathcal{A}\} = \left(\bigcap\{A^+ \mid A \in \mathcal{A}\}\right) \cap \left(\bigcap\{B^- \mid B \in \perp X(\mathcal{A})\}\right). \quad \square$$

Proposition 2.1.15. *1) The map $\perp X$ is an involution, i. e. $\perp X \circ \perp X = 1_{GX}$.*

2) $\perp = (\perp X) \colon GX \to GX$ is a functorial isomorphism.

3) $\lambda X = \{\mathcal{A} \in GX \mid \mathcal{A} = \perp X(\mathcal{A})\}$.

Proof. 1) Obviously, $(\perp X \circ \perp X)(\mathcal{A}) \supset \mathcal{A}$. To prove the inclusion $\subset$, assume the contrary. Then, for every $A \in \mathcal{A}$ we have $A \cap (X \setminus C) \neq \varnothing$.

Prove that there is a closed subspace D of X such that $D \subset X \setminus C$ and $A \cap D \neq \varnothing$. Indeed, otherwise for every such D choose $A_D \in \mathcal{A}$, $A_D \cap D = \varnothing$. We obtain a net $(A_D \mid D \in \exp X,\ D \subset X \setminus C)$ ordered

by inclusion of indices ($D \le D'$ iff $D \subset D'$). It is easy to see that for any limit point A_0 of this net we have $A_0 \subset C$. Then $A_0 \in \mathcal{A}$ and, consequently, $C \in \mathcal{A}$.

We have just shown that there is $D_0 \in \exp X$ such that $D_0 \cap C = \varnothing$ and $D_0 \cap A \neq \varnothing$ for every $A \in \mathcal{A}$. Thus, $D_0 \in \perp X(\mathcal{A})$ and we obtain a contradiction with the fact that $C \in (\perp X \circ \perp X)((\mathcal{A})))$.

2) Obvious.

3) If $\mathcal{M} \in \lambda X$ and $A \in \perp X(\mathcal{M})$, then $A \in \mathcal{M}$, by maximality of $\mathcal{M}$. The inclusion $\mathcal{M} \subset \perp X(\mathcal{M})$ is obvious. Thus, $\mathcal{M} = \perp X(\mathcal{M})$ for every $\mathcal{M} \in \lambda X$.

Now if $\mathcal{M} = \perp X(\mathcal{M})$, then $\mathcal{M}$ is linked. To see that $\mathcal{M}$ is a maximal linked system, consider $A \in \exp X$ such that $A \cap M \neq \varnothing$ for every $M \in \mathcal{M}$. But then $A \in \perp X(\mathcal{M}) = \mathcal{M}$. □

Exercises

1. Define a ternary operation $\mu\colon \lambda X^3 \to \lambda X$ by the formula $\mu(\mathcal{L}, \mathcal{M}, \mathcal{N}) = (\mathcal{L} \cap \mathcal{M}) \cup (\mathcal{M} \cup \mathcal{N}) \cup (\mathcal{L} \cup \mathcal{M})$. Prove that μ is a *mixer*, i. e. μ is invariant with respect to any permutation of its arguments, and $\mu(x, x, y) = x$.
2. Prove that for a map $f\colon X \to Y$ we have $Gf = rY \circ \exp^2 f|GX\colon GX \to GY$.
3. Prove that $\perp X(\mathcal{A} \cup \mathcal{B}) = \perp X(\mathcal{A}) \cap \perp X(\mathcal{B})$, $\perp X(\mathcal{A} \cap \mathcal{B}) = \perp X(\mathcal{A}) \cap \perp X(\mathcal{B})$ for every $\mathcal{A}, \mathcal{B} \in GX$. More generally, prove that $\perp X(\cup\mathfrak{A}) = \cap \perp X(\mathfrak{A})$, $\perp X(\cap\mathfrak{A}) = \cup \perp X(\mathfrak{A})$, for every nonempty closed subset $\mathfrak{A} \subset GX$.
4. Find an internal description of the elements of the space $\perp X(NX)$.
5. Let (X, d) be a compact metric space. Show that the function $d_V\colon \lambda X \times \lambda X \to \mathbb{R}$, $d_V(\mathcal{M}, \mathcal{N}) = \inf\{\varepsilon > 0 \mid$ for every $M \in \mathcal{M}$ there exists $N \in \mathcal{N},\ N \in O_\varepsilon(M)\}$, is a compatible metric on λX.

2.1.4. Functors $\widetilde{G}, \widetilde{N}_k$

For a compact Hausdorff space X let

$$\widetilde{G}X = \{\mathcal{A} \in \exp^2 \mid B \in \mathcal{A} \text{ whenever } B \in \exp X,\ B \supset A \text{ for some } A \in \mathcal{A}, \text{ and } B \subset \bigcup\mathcal{A}\}.$$

We endow $\widetilde{G}X$ with the inherited from $\exp^2 X$ topology. It is easy to verify that the map $\gamma X\colon \exp^2 X \to \widetilde{G}X$, defined by

$$\gamma X(\mathcal{A}) = \{B \in \exp X \mid B \subset \bigcup\mathcal{A} \text{ and there exists } A \in \mathcal{A} \text{ with } B \supset A\},$$

is a continuous retraction. Hence, $\widetilde{G}X$ is a compact Hausdorff space, and $\widetilde{G}$ is a subfunctor of $\exp^2$.

Denote by $\widetilde{N}_k X$ the subspace of $\widetilde{G}X$ formed by all k-linked systems containing in $\widetilde{G}X$. We obtain the subfunctor $\widetilde{N}_k$ of $\widetilde{G}$.

Define the subfunctor $\tilde{\lambda}$ of the functor $\widetilde{G}$ by

$$\tilde{\lambda}X = \{\mathcal{A} \in \widetilde{G}X \mid \mathcal{A} \in \lambda(\cup\mathcal{A})\}.$$

Exercises

1. Show that $\gamma = (\gamma X)\colon \exp^2 \to \widetilde{G}$ is a natural transformation.
2. Suppose that ϱ is a metric on X and denote by ϱ_{HH} the corresponding Hausdorff metric on $\exp^2 X$. Prove that the retraction $\gamma X\colon \exp^2 X \to \widetilde{G}X$ is a nonexpanding map with respect to the metric ϱ_{HH}.
3. Define the map $\bar{\perp} X\colon \widetilde{G}X \to \widetilde{G}X$ by $\bar{\perp} X(\mathcal{A}) = \{B \in \exp X \mid B \subset \cup\mathcal{A}$ and $B \cap A \neq \varnothing$ for every $A \in \mathcal{A}\}$. Prove that the map $\bar{\perp} X$ is continuous. Prove the counterpart of Proposition 2.1.15 for $\bar{\perp}$.

2.1.5. Probability measure functor and related functors

For a compact Hausdorff space X by $C(X)$ we denote the space of continuous real-valued functions endowed with the standard sup-norm:

$$||f|| = \sup\{|\varphi(x)| \mid x \in X\}, \ \varphi \in C(X).$$

The set $C(X)$ has a natural partial order $\leq$: $\varphi \leq \psi$, whenever $\varphi(x) \leq \psi(x)$ for every $x \in X$. Let $M(X)$ denote the adjoint to $C(X)$ space, i.e. the space of continuous linear functionals on $C(X)$. A base of the *$*$-weak topology* on $M(X)$ consists of the sets of the form

$$O(\mu;\varphi_1,\ldots,\varphi_k;\varepsilon) = \{\mu' \in M(X) \mid |\mu(\varphi_i) - \mu'(\varphi_i)| < \varepsilon,\ i = 1,\ldots,k\}, \tag{2.2}$$

where $\mu \in M(X)$, $\varphi_1,\ldots,\varphi_k \in C(X)$, $\varepsilon > 0$.

A functional $\mu \in M(X)$ is called *positive,* whenever $\mu(\varphi) \geq 0$ for every $\varphi \geq 0$. A functional $\mu \in M(X)$ is called *regular* if

$$||\mu|| = \sup\{|\mu(\varphi)| \mid \varphi \in C(X),\ ||\varphi|| \leq 1\} = 1.$$

A regular positive functional is called a *probability measure*. We denote by PX the space of all probability measures on X.

The diagonal map $\mu \mapsto (\mu(\varphi))_{\varphi\in C(X)}$ is a closed embedding of PX into a compact Hausdorff space $\prod_{\varphi\in C(X)}[-||\varphi||, ||\varphi||]$. Thus, $PX \in$ $|\mathbf{Comp}|$.

Every map $f\colon X \to Y$ induces a map $Pf\colon PX \to PY$ by the formula

$$Pf(\mu)(\varphi) = \mu(\varphi \circ f), \quad \varphi \in C(Y).$$

Proposition 2.1.16. *Let $f\colon X \to Y$ be a Milutin map of compact Hausdorff spaces. Then there exists an affine map $s\colon PY \to PX$ such that $Pf \circ s = 1_{PY}$.*

Proof. Let $u\colon C(X) \to C(Y)$ be an averaging operator for f. For every $\mu \in PY$ let $s(\mu)(\varphi) = \mu(u(\varphi))$, $\varphi \in C(X)$. It is easy to prove that $s(\mu) \in PX$ and s is a required map. □

Let $x \in X$. The functional $\delta_x \in M(X)$, $\delta_x(\varphi) = \varphi(x)$, is a probability measure on X (the *Dirac measure* concentrated in x).

Note that the space PX is closed under convex combinations. In particular, given $\alpha_1, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ and $x_1, \dots, x_n \in X$ we can define the probability measure $\sum_{i=1}^n \alpha_i \delta_{x_i}$.

Let $\alpha_i \geq 0$, $i \in \mathbb{N}$, and $\sum_{i=1}^\infty \alpha_i = 1$. The measure of the form $\mu = \sum_{i=1}^\infty \alpha_i \delta_{x_i}$, $x_i \in X$, is called *atomic.*

Let $\mu \in PX$. The *support* of μ is the minimal (with respect to inclusions) closed subset A of X with the property: $\mu(\varphi) = 0$ for every $\varphi \in C(X)$ with $\varphi|A \equiv 0$. The support of μ is denoted by $\operatorname{supp}\mu$. If $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$ and $\alpha_i > 0$ for every $i = 1, \dots, n$, then $\operatorname{supp}_\mu = \{x_1, \dots, x_n\}$.

Given $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i} \in PX$, and $\nu = \sum_{j=1}^m \beta_j \delta_{y_j} \in PY$ let

$$\mu \otimes \nu = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \delta_{(x_i, y_j)}.$$

The probability measure $\mu \otimes \nu \in P(X \times Y)$ is called the *tensor product* of μ and ν.

Every probability measure $\mu \in PX$ can be also considered as a function defined on closed subsets of X: given $A \in \exp X$ let

$$\mu(A) = \inf\{\mu(\varphi) \mid \varphi \in C(X, [0,1]),\ \varphi|A \equiv 1\}.$$

The number $\mu(A)$ is called the *measure* of A. If U is an open subset of X, then the measure of U is defined as $\mu(U) = 1 - \mu(X \setminus U)$.

In fact, it is possible to define the restriction $\mu|A$ by the following manner (we endow the set $\{\psi \in C(X, [0,1]),\ \varphi|A \equiv 1\}$ with the reverse pointwise order relation) :

$$\mu|A(\varphi) = \lim\{\mu(\varphi \cdot \psi) \mid \psi \in C(X, [0,1]),\ \varphi|A \equiv 1\}.$$

Let $\varphi \in C(X)$ and $\mu \in PX$. By $\varphi\mu$ we denote the (not necessarily probability) measure defined by the formula $\varphi\mu(\psi) = \mu(\varphi \cdot \psi)$ (the *product of a measure and a function*).

Note that PX is a convex subspace of the locally convex space $M(X)$. Thus, the space $\mathrm{cc}\,PX$ is defined. The functor $\mathrm{cc}\,P\colon \mathbf{Comp} \to \mathbf{Comp}$ is naturally defined.

Besides, we are able to define the functor $G_{\mathrm{cc}}P$ by the following manner. Let

$$\begin{aligned} G_{\mathrm{cc}}PX = \{\mathcal{A} \mid &\mathcal{A} \text{ is a closed subset of } \mathrm{cc}\,PX \text{ and } A \in \mathcal{A}, \\ &A \subset B \in \mathrm{cc}\,PX \implies B \in \mathcal{A}\}. \end{aligned}$$

For a map $f\colon X \to Y$ in $\mathbf{Comp}$ and $\mathcal{A} \in G_{\mathrm{cc}}PX$ let

$$G_{\mathrm{cc}}PF(\mathcal{A}) = \{B \in G_{\mathrm{cc}}PY \mid Pf(\mathcal{A}) \subset B\}.$$

It is easy to show that $G_{\mathrm{cc}}P$ is an endofunctor in $\mathbf{Comp}$.

2.1.6. Functor of order-preserving functionals

For each $c \in \mathbb{R}$ by c_X we denote the constant function from $C(X)$ defined by the formula $c_X(x) = c$ for each $x \in X$.

A functional (which is not supposed a priori to be either linear or continuous) $\nu\colon C(X) \to \mathbb{R}$ is called *weakly additive* if for each $c \in \mathbb{R}$ and $\varphi \in C(X)$ we have $\nu(\varphi + c_X) = \nu(\varphi) + c$; *order-preserving* if for each $\varphi, \psi \in C(X)$ with $\varphi \le \psi$ we have $\nu(\varphi) \le \nu(\psi)$.

The space of real numbers $\mathbb{R}$ is endowed with the standard metric.

Lemma 2.1.17. *Each order-preserving weakly additive functional is a non-expanding map.*

Proof. Let $\nu\colon C(X) \to \mathbb{R}$ be an order-preserving weakly-additive functional and $\varphi, \psi \in C(X)$. If $\|\varphi - \psi\| = a \in \mathbb{R}$, then $\varphi - a_X \le \psi \le \varphi + a_X$ and $\nu(\varphi) - a \le \nu(\psi) \le \nu(\varphi) + a$. Thus $|\nu(\varphi) - \nu(\psi)| \le a$. □

Corollary 2.1.18. *Each order-preserving weakly additive functional is continuous.*

A subset $L \subset C(X)$ is called an *A-subspace* if $0_X \in L$ and for each $\varphi \in L$, $c \in \mathbb{R}$ we have $\varphi + c_X \in L$.

Lemma 2.1.19. *For each A-subspace $L \subset C(X)$ and for each order-preserving weakly additive functional $\nu\colon L \to \mathbb{R}$ there exists an order-preserving weakly additive functional $\nu'\colon C(X) \to \mathbb{R}$ such that $\nu'|L = \nu$.*

Proof. Let us consider the set of all pairs (B, μ), where $L \subset B \subset C(X)$ is an A-space and μ is an order-preserving weakly additive functional. This set can be regarded as a partially ordered set by the order $(B_1, \mu_1) \le (B_2, \mu_2)$ iff $B_1 \subset B_2$ and μ_2 is an extension of μ_1. By the Zorn Lemma, there exists a maximal element (B_0, μ_0).

Suppose that $B_0 \ne C(X)$. Take any $\varphi \in C(X) \setminus B_0$. Let B^+ (B^-) be the set of all $\psi \in B_0$ with $\psi \ge \varphi$ (respectively $\psi \le \varphi$). Then we can choose $p \in \mathbb{R}$ with $\mu_0(B^-) \le p \le \mu_0(B^+)$. The set $D = B_0 \cup \{\varphi + c_X \mid c \in \mathbb{R}\}$ is an A-subset in $C(X)$. Define the functional $\mu\colon D \to \mathbb{R}$ as follows: $\mu|B_0 = \mu_0$ and $\mu(\varphi + c_X) = p + c$, $c \in \mathbb{R}$. It is easy to check that μ is an order-preserving weakly additive functional and we obtain the contradiction with the maximality of (B_0, μ_0). □

For $X \in |\mathbf{Comp}|$ we denote by OX the set of all order-preserving weakly additive normed functionals in $C(X)$. It is easy to see that for each $\nu \in OX$ and $c \in \mathbb{R}$ we have $\nu(c_X) = c$.

We consider OX as the subspace of the space $C_p(C(X))$ of all continuous functions on $C(X)$ equipped with the point-wise convergence topology. The base of this topology consist of sets of the form (2.2).

Proposition 2.1.20. *If $X \in |\mathbf{Comp}|$, so is OX.*

Proof. First, observe that OX is contained in the Tychonov product of closed intervals $Z = \prod\{[-\|\varphi\|, \|\varphi\|] \mid \varphi \in C(X)\}$. Thus it is sufficient to prove that OX is closed in Z.

Consider $\mu \in Z \setminus OX$. Then μ fails to satisfy one of the three condition from the definition of OX.

Suppose μ is not normed. Then we have $(\mu; 1_X; \frac{|\mu(1_X)-1|}{2}) \cap OX = \varnothing$.

Suppose μ is not weakly additive. Then there exist $\varphi \in C(X)$ and $c \in \mathbb{R}$ such that $\mu(\varphi + c_X) \ne \mu(\varphi) + c$. Put $\delta = |\mu(\varphi + c_X) - \mu(\varphi) - c| > 0$. Then $(\mu; \varphi + c_X, \varphi, c_X, \delta/4) \cap OX = \varnothing$.

Finally, suppose μ is not order-preserving. Then there exist $\varphi_1, \varphi_2 \in C(X)$ such that $\varphi_1 \ge \varphi_2$ and $\mu(\varphi_1) < \mu(\varphi_2)$. Put $\varepsilon = \mu(\varphi_2) - \mu(\varphi_1)$. Then $(\mu; \varphi_1, \varphi_2; \varepsilon/2) \cap OX = \varnothing$. Thus OX is a closed subset of Z. □

Let us remark that the space OX is naturally embedded in the product $\prod_{\varphi \in C(X)} \mathbb{R}_\varphi$ which is a locally convex topological linear space and a topological partial ordered space with respect to the coordinate-wise order.

Let $f\colon X \to Y$ be a morphism in **Comp**. Define the map $Of\colon OX \to OY$ by the formula $(Of(\mu))(\varphi) = \mu(\varphi \circ f)$, where $\mu \in OX$ and $\varphi \in C(Y)$.

It is easy to check that Of is well-defined, continuous and $O(f \circ g) = Of \circ Og$. Thus O is a covariant functor in the category **Comp**.

2.1.7. G-symmetric power functor

Let G be a subgroup of the n-symmetric group S_n. Recall that SP^n_G denotes the *G-symmetric power functor* defined as follows. For a space X the space $SP^n_G X$ is the orbit space of the n-th power X^n by the permutations:

$$(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

where $\sigma \in G$.

The orbit containing $(x_1, \dots, x_n)$ is denoted by $[x_1, \dots, x_n]_G$. The set $\{x_1, \dots, x_n\}$ is called the support of an element $[x_1, \dots, x_n]_G$ and is denoted by $\mathrm{supp}([x_1, \dots, x_n]_G)$. For a map $f\colon X \to Y$ the map $SP^n_G f\colon SP^n_G X \to SP^n_G Y$ is defined as follows:

$$SP^n_G f[x_1, \dots, x_n]_G = [f(x_1), \dots, f(x_n)]_G.$$

If $G = S_n$, we abbreviate the denotation of SP^n_G to SP^n (the symmetric power functor).

For the case $G = \{e\}$ we obtain the *power functor* $(-)^n$.

Proposition 2.1.21. *The maps $\varphi X\colon SP^n X \to PX$,*

$$\varphi X[x_0, \dots, x_{n-1}] = \frac{1}{n} \sum_{i \in n} \delta_{x_i},$$

form a natural transformation of SP^n to P.

Proof. Obvious. □

The G-symmetric power functors are also referred as the *Borsuk-Ulam functors*.

2.2. Some elementary properties of functors

A functor is called *monomorphic* (respectively, *epimorphic*) provided it preserves monomorphisms (respectively epimorphisms). For a monomorphic functor F in **Comp**, spaces $A, X \in |\mathbf{Comp}|$ with $A \subset X$ we always identify the space FA with a subspace in FX along the embedding Fi, where $i\colon A \to X$ is the inclusion map.

A monomorphic functor F is said to be *preimage-preserving* if it preserves the universality of the diagrams of the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \arrow[r, "i"] & Y, \end{tikzcd}$$

where i is a monomorphism. Equivalently, $F\colon \mathbf{Comp} \to \mathbf{Comp}$ preserves preimages if $F\big(f^{-1}(A)\big) = (Ff)^{-1}(FA)$ for every closed subset A of Y.

Let F be a monomorphic functor. We say that F *preserves (finite) intersection* whenever

$$F\big(\bigcap\{A_\alpha \mid \alpha \in \Gamma\}\big) = \bigcap\{FA_\alpha \mid \alpha \in \Gamma\}$$

for every (finite) family of closed subsets $\{A_\alpha \mid \alpha \in \Gamma\}$ in X.

An endofunctor F in **Comp** is called *continuous* if it preserves the limits of inverse systems $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ over directed sets $\mathcal{A}$.

For a functor F and a compact Hausdorff space X denote by $F_\omega X$ the subspace $\bigcup\{Ff(Fn) \mid f \in C(n, X),\ n \in \mathbb{N}\}$ of FX.

Proposition 2.2.1. *Let F be a continuous functor. Then F is epimorphic iff $F_\omega X$ is a dense subset of FX for every compact Hausdorff space X.*

Proof. Show only the necessary part. Let X be a compact Hausdorff space. Consider a continuous surjection $f\colon Z \to X$ of a zero-dimensional compact Hausdorff space Z. Since, $Ff(F_\omega Z) = F_\omega X$, it is sufficient to prove that $F_\omega Z$ is dense in FZ. Consider an inverse system $\mathcal{S} = \{Z_\alpha, f_{\alpha\beta}; \mathcal{A}\}$ of finite compacta Z_α with a limit Z. For every limit map $f_\alpha\colon Z \to Z_\alpha$ we have $Ff_\alpha(F_\omega Z) = FZ_\alpha$ by continuity of F. And this yields a desirable property of $F_\omega Z$. $\square$

Proposition 2.2.2. *Let F be a monomorphic and epimorphic endofunctor in* **Comp**. *Then F is continuous if and only if for every cardinal number τ a family $Fp_A\colon F(I^\tau) \to F(I^A)$, $|A| < \omega$, separates points (here $p_A\colon I^\tau \to I^A$ is a projection).*

Proof. Consider the inverse system $\mathcal{S} = \{I^A, p_{AB}; \mathcal{P}_\omega(\tau)\}$ and denote by $h\colon FI^\tau \to F\varprojlim \mathcal{S}$ the canonical map, $h = (Fp_A)_{A\in\mathcal{P}_\omega(\tau)}$. Clearly, h is one-to-one.

Since F is epimorphic, for any $(a_A)_{A\in\mathcal{P}_\omega(\tau)}$ there exist elements b_A in $F(I^\tau)$ such that $Fp_A(b_A) = a_A$, $A \in \mathcal{P}_\omega(\tau)$. For a limit point b of the net $(a_\alpha)_{A\in\mathcal{P}_\omega(\tau)}$ we have $h(b) = (a_A)_{A\in\mathcal{P}_\omega(\tau)}$. Thus, we have proved continuity of F on the inverse system consisting of the faces of the Tychonov cubes. Since every inverse system can be embedded in such a system, we are done. $\square$

Theorem 2.2.3. *A monomorphic and epimorphic functor F in the category* **Comp** *is continuous iff the map $F\colon C(X,Y) \to C(FX, FY)$, $F(f) = Ff$, is continuous for every compact Hausdorff spaces X and Y.*

Proof. Necessity. Let $A_n = \{k \mid 1 \le k < n\}$ and $\alpha\mathbb{N} = \mathbb{N} \cup \{\infty\}$ be an Aleksanrov compactification of $\mathbb{N}$. Consider the following maps: $i_n\colon X \to X \times \alpha\mathbb{N}$, $n \in \alpha\mathbb{N}$, $i_n(x) = (x, n)$, $x \in X$, and $\pi^m_n\colon X \times A_m \to X \times A_n$, $m \ge n$, $\pi^m_n(x,k) = (x, \min\{k, n\})$. The space $X \times \alpha\mathbb{N}$ is naturally identified with the limit of an inverse system $\mathcal{S} = \{X \times A_n, \pi^m_n\}$. Denote by $\pi_n\colon X \times \alpha\mathbb{N} \to X \times A_n$, $n \in \mathbb{N}$, limit projections of $\mathcal{S}$.

Show that $\lim_{n\to\infty} Fi_n = Fi_\infty$ in $C(FX, F(X \times \alpha\mathbb{N}))$. By continuity of the functors F and $C(X, -)$ (see Exercise 1) we have

$$C(FX, F(X \times \alpha\mathbb{N})) = \varprojlim\{C(FX, F(X \times A_n)), F\pi^m_n\}.$$

Therefore, it is sufficient to prove that

$$\lim_{n\to\infty} F\pi_k \circ Fi_n = F\pi_k \circ Fi_\infty$$

for every $k < \infty$. This fact follows from equalities $\pi_k \circ i_n = \pi_k \circ i_k$ for all $k \ge n \ge \infty$.

If Y is a compact metrizable space then a space $C(X,Y)$ is also metrizable. Thus we have only to prove the continuity of map $f \mapsto Ff$ on countable sequences in this case. Let a sequence $\{f_n\}$ tends to f_∞ in

$C(X,Y)$. Define a map $\Phi\colon X\times\alpha\mathbb{N}\to Y$ by the formula $\Phi(x,n)=f_n(x)$, $n\le\infty$. Evidently, it is continuous. By the previous paragraphs

$$\lim_{n\to\infty} Ff_n = \lim_{n\to\infty} F\Phi\circ Fi_n = F\Phi\circ Fi_\infty = Ff_\infty.$$

If a compact Hausdorff space Y is non-metrizable, present it as a limit of a continuous system of compact metrizable spaces: $Y=\varprojlim\{Y_\alpha,p_{\alpha\beta}\}$. By continuity of the functor $C(X,-)$ we have $C(X,Y)=\varprojlim C(X,Y_\alpha)$ and $C(FX,FY)=\varprojlim C(FX,FY_\alpha)$. Being the limit map of a system morphism formed by continuous maps $C(X,Y_\alpha)\to C(FX,FY_\alpha)$, the map $f\mapsto Ff$ is continuous.

Sufficiency. Let $X=Y=I^\tau$. For every $A\in\mathcal{P}_\omega(\tau)$ let $f^A=(f^A_\alpha)_{\alpha\in\tau}\in C(I^\tau,I^\tau)$ be a map,

$$f^A_\alpha((x_\gamma)_{\gamma\in\tau})=\begin{cases}x_\alpha, & \text{if } \alpha\in A,\\ 0, & \text{otherwise.}\end{cases}$$

Then the net $(f_A)_A\in\mathcal{P}_\omega(\tau)$ converges to 1_{I^τ}. Thus $(Ff_A)_A\in\mathcal{P}_\omega(\tau)$ converges to $F1_{I^\tau}=1_{F(I^\tau)}$, and, therefore, the family $\{Ff_A)\mid A\in\mathcal{P}_\omega(\tau)\}$ separates points. Now we have to apply Proposition 2.2.2. □

Proposition 2.2.4. *If a functor F preserves points, then a natural transformation φ: Id $\to F$ is unique.*

Proof. Let $i_x\colon\{x\}\to X$ be the identity embedding. Then the following diagram

$$\begin{array}{ccc}\{x\} & \xrightarrow{\varphi\{x\}} & F\{x\}\\ {\scriptstyle i_x}\downarrow & & \downarrow{\scriptstyle Fi_x}\\ X & \xrightarrow{\varphi X} & FX\end{array}$$

is commutative. Since F preserves points, the map $\varphi\{x\}$ is uniquely determined. Therefore, the point $\varphi X(x)=Fi_x\circ\psi\{x\}\circ i_x^{-1}(x)$ is determined indepedently of the choice of the transformation φ. □

For the functor F and compacta X and Y there exists a unique map

$j_{FX,Y}\colon FX\times Y\to F(X\times Y)$ such that for every $y\in Y$ the diagram

$$\begin{tikzcd} FX\times\{y\} \arrow[r, "\mathrm{id}_{FX}\times i_y"] \arrow[d, "h_y"'] & FX\times Y \arrow[d, "j_{FX,Y}"] \\ F(X\times\{y\}) \arrow[r, "F(\mathrm{id}_X\times i_y)"] & F(X\times Y) \end{tikzcd} \tag{2.3}$$

is commutative, where $i_y\colon\{y\}\to Y$ is the embedding and h_y is the natural homeomorphism, defined as the composition

$$FX\times\{y\}\xrightarrow{\mathrm{pr}_1}FX\xrightarrow{Fg_y}F(X\times\{y\}),$$

in which $g_y(x)=(x,y)$. Indeed, on each "horisontal" fiber $FX\times\{y\}$ of the product $FX\times Y$ the map $j_{FX,Y}$ coincides with the composition $F(\mathrm{id}_X\times i_y)\circ h_y$.

Proposition 2.2.5. *For every continuous map $f\colon Y\to Z$ the diagram*

$$\begin{tikzcd} FX\times Y \arrow[r, "\mathrm{id}_{FX}\times f"] \arrow[d, "j_{FX,Y}"'] & FX\times Z \arrow[d, "j_{FX,Z}"] \\ F(X\times Y) \arrow[r, "F(\mathrm{id}_X\times f)"] & F(X\times Z) \end{tikzcd} \tag{2.4}$$

is commutative.

Proof. Let $a\in FX$ and $y\in Y$. Then the commutativity of diagram (2.3) implies that $j_{FX,Y}(a,y)=F(\mathrm{id}_X\times i_y)\circ h_y(a,y)$. Hence,

$$\begin{aligned} F(\mathrm{id}_X\times f)\circ j_{FX,Y}(a,y)&=F(\mathrm{id}_X\times f)\circ F(\mathrm{id}_X\times i_y)\circ h_y(a,y)=\\ &=F(\mathrm{id}_X\times(f\circ i_y))\circ h_y(a,y).\end{aligned}$$

On the other hand,

$$j_{FX,Z}(a,f(y))=F(\mathrm{id}_X\times i_{f(y)})\circ h_{f(y)}(a,f(y)).$$

Therefore, the commutativity of diagram (2.4) follows from the obvious commutativity of the diagram

$$\begin{tikzcd} FX\times\{y\} \arrow[r] \arrow[d, "h_y"'] & FX\times\{f(y)\} \arrow[d, "h_{f(y)}"'] \\ F(X\times\{y\}) \arrow[r] & F(X\times\{f(y)\}). \end{tikzcd}$$

$\square$

Proposition 2.2.6. *If a functor F is continuous and preserves points, then the map $j_{FX,Y}$ is continuous.*

Proof. Assume first that we can prove that the map $j_{FX,Y}$ is continuous for zero-dimensional Y. Then for an arbitrary compact Hausdorff space Z there exist a zero-dimensional compact Hausdorff space Y and an epimorphism $f\colon Y \to Z$. The map $j_{FX,Z}$ in the commutative diagram (2.4) is a left divisor of the continuous map $j_{FX,Z} \circ \mathrm{id}_{FX} \times f = F(\mathrm{id}_X \times f) \circ j_{FX,Y}$, while the right divisor $\mathrm{id}_{FX} \times f$ of this map is an epimorphism. Under such conditions the map $j_{FX,Z}$ is continuous.

We now prove that $j_{FX,Y}$ is continuous for zero-dimensional Y. Let Y be represented as the limit of an inverse system $\mathcal{S} = \{Z_\alpha, \pi^\alpha_\beta; \alpha \in \mathcal{A}\}$ of finite spaces. Then the compact Hausdorff space $FX \times Y$ is the limit of the inverse system $FX \times \mathcal{S} = \{FX \times Z_\alpha, \mathrm{id}_{FX} \times \pi^\alpha_\beta; \alpha \in \mathcal{A}\}$. By the continuity of the functor F, the compact Hausdorff space $F(X \times Y)$ is the limit of the inverse system $F(X \times \mathcal{S}) = \{F(X \times Z_\alpha), F(\mathrm{id}_X \times \pi^\alpha_\beta l); \alpha \in \mathcal{A}\}$. Further, according to Proposition 2.2.5, the maps $\{j_{FX,Z_\alpha}; \alpha \in \mathcal{A}\}$, which are clearly continuous for the finite spaces Z_α, form a morphism $\mathcal{J}$ of the system $FX \times \mathcal{S}$ into the system $F(X \times \mathcal{S})$. The limit map $\varprojlim \mathcal{J}\colon FX \times Y \to F(X \times Y)$ is defined. It remains to verify that $\varprojlim \mathcal{J} = j_{FX,Y}$ Since $j_{FX,Y}$ is unique, for this it suffices to verify that diagram (2.3) is still commutative if we replace the map $j_{FX,Y}$ by $\varprojlim \mathcal{J}$. Let Z_α be an element of the system $\mathcal{S}$, and let $y \in Y$.

There exists an embedding $e^\alpha\colon Z_\alpha \to Y$ such that $e^\alpha(\pi_\alpha(y)) = y$ and $\pi_\alpha \circ e^\alpha = \mathrm{id}_{Z_\alpha}$ (here π_α is a limit map of $\mathcal{S}$). Then the diagram

$$\begin{tikzcd}
FX \times Y \arrow[d, "\varprojlim \mathcal{J}"'] & FX \times Z_\alpha \arrow[l, "\mathrm{id}_{FX} \times e^\alpha"'] \arrow[d, "j_{FX,Z_\alpha}"] \\
F(X \times Y) & F(X \times Z_\alpha) \arrow[l, "F(\mathrm{id}_X \times e^\alpha)"']
\end{tikzcd} \tag{2.5}$$

is commutative. Indeed, let $\alpha < \beta \in \mathcal{A}$ and $e^\alpha_\beta = \pi_\beta \circ e^\alpha$. Then by Proposition 2.2.5, the diagram

$$\begin{tikzcd}
FX \times Z_\beta \arrow[d, "j_{FX,Z_\beta}"'] & FX \times Z_\alpha \arrow[l, "\mathrm{id}_{FX} \times e^\alpha_\beta"'] \arrow[d, "j_{FX,Z_\alpha}"] \\
F(X \times Z_\beta) & F(X \times Z_\alpha) \arrow[l, "F(\mathrm{id}_X \times e^\alpha_\beta)"']
\end{tikzcd} \tag{2.6}$$

is commutative. Therefore, diagram (2.5) is commutative, being the limit of the commutative diagrams in (2.6).

Denote by $\pi_\alpha^y \colon \{y\} \to Z_\alpha$ the restriction of the map π_α to the singleton subspace $\{y\} \subset Y$. Then, since the diagram

$$\begin{tikzcd} & \{y\} \arrow[dl, "i_y"'] \arrow[dr, "\pi_\alpha^y"] & \\ Y & & Z_\alpha \arrow[ll, "e^\alpha"'] \end{tikzcd}$$

is commutative, so are the diagrams

$$\begin{tikzcd} & FX \times \{y\} \arrow[dl, "\mathrm{id}_{FX} \times i_y"'] \arrow[dr, "\mathrm{id}_{FX} \times \pi_\alpha^y"] & \\ FX \times Y & & FX \times Z_\alpha \arrow[ll, "\mathrm{id}_{FX} \times e^\alpha"'] \end{tikzcd}$$

and

$$\begin{tikzcd} & F(X \times \{y\}) \arrow[dl, "F(\mathrm{id}_X \times i_y)"'] \arrow[dr, "F(\mathrm{id}_X \times \pi_\alpha^y)"] & \\ F(X \times Y) & & F(X \times Z_\alpha). \arrow[ll, "F(\mathrm{id}_X \times e^\alpha)"'] \end{tikzcd}$$

Together with the commutativity of diagrams (2.5) and

$$\begin{tikzcd} FX \times \{y\} \arrow[r, "\mathrm{id}_{FX} \times \pi_\alpha^y"] \arrow[d, "h_y"'] & FX \times Z_\alpha \arrow[d, "j_{FX, Z_\alpha}"] \\ F(X \times \{y\}) \arrow[r, "F(\mathrm{id}_X \times \pi_\alpha^y)"] & F(X \times Z_\alpha) \end{tikzcd}$$

(the commutativity of the latter diagram follows both from the commutativity of (2.3) and from Proposition (2.2.5), because it is obvious that $h_y = j_{FX,\{y\}}$), this gives us the commutativity of the diagram

$$\begin{tikzcd} FX \times \{y\} \arrow[r, "\mathrm{id}_{FX} \times i_y"] \arrow[d, "h_y"'] & FX \times Z \arrow[d, "\varprojlim \mathcal{J}"] \\ F(X \times \{y\}) \arrow[r, "F(\mathrm{id}_X \times i_y)"] & F(X \times Y). \end{tikzcd}$$

$\square$

Proposition 2.2.7. *For a point-preserving continuous functor F there exists a unique natural transformation η: $\mathrm{Id} \to F$. The map $\eta X : X \to FX$ is defined as the composition*

$$X \longrightarrow F1 \times X \xrightarrow{j_{F1,X}} F(1 \times X) \longrightarrow FX.$$

Proof. It is an obvious corollary of Propositions 2.2.4 and 2.2.5. □

Let F be an endofunctor in **Comp**. The cardinal number $\mathrm{w}(F) = \sup\{\mathrm{w}(n) \mid n \in \mathbb{N}\}$ is called the *weight* of F.

Proposition 2.2.8. *Suppose F is monomorphic, epimorphic, continuous endofunctor in* **Comp**. *Then for every $X \in |$***Comp***$|$ with $\mathrm{w}(X) \geq \mathrm{w}(F)$ we have $\mathrm{w}(FX) \leq \mathrm{w}(X)$.*

Proof. First, we show that $\mathrm{w}(F(2^\omega)) = \mathrm{w}(F)$. Since $2^\omega = \varprojlim 2^n$ and F is continuous, the space $F(2^\omega)$ is homeomorphic to $\varprojlim F(2^n)$. Thus,

$$\mathrm{w}(F(2^\omega)) = \sup\{\mathrm{w}(F(2^n)) = \mathrm{w}(F)$$

(note that each 2^n is embeddable in 2^ω and F is monomorphic). Since every compact metrizable space X is a continuous image of 2^ω and F is epimorphic, we see that $\mathrm{w}(FX) \leq \mathrm{w}(F(2^\omega))$.

Now let I^τ be the Tychonov cube of weight τ. Then $I^\tau = \varprojlim I^A$, where I^A are the finite-dimensional cubes. Then we obviously obtain $\mathrm{w}(F(I^\tau)) \leq \max\{\tau, \mathrm{w}(F)\}$.

Finally, given $X \in |\mathbf{Comp}|$ with $\mathrm{w}(X) \geq \mathrm{w}(F)$, embed X into the Tychonov cube $I^{\mathrm{w}(X)}$, then

$$\mathrm{w}(FX) \leq \mathrm{w}(F(I^{\mathrm{w}(X)})) \leq \max\{\mathrm{w}(X), \mathrm{w}(F)\} = \mathrm{w}(X).$$

□

Exercises

1. Prove the continuity of the functor $C(X, -)$: **Tych** $\to$ **Tych**.
2. (See Nykyforchyn [1999].) Prove that the continuity of a monomorphic and epimorphic functor in **Comp** is equivalent to the preservation of intersections of decreasing families of subsets.
3. Show that there is a natural embedding $(-)^n \to P$.
4. Describe the subgroups G of the symmetric group S_n for which there exists an embedding $SP^n_G \to P$.

2.3. Definition of normal functor

Definition 2.3.1. An endofunctor F in **Comp** is called *normal* if it is continuous, monomorphic, epimorphic, preserves weight of infinite compacta, intersections, preimages, singletons and empty set.

A functor F is said to be *weakly normal (almost normal)* if it satisfies all the properties from the previous definition excepting perhaps the preimage preserving property (respectively, epimorphness). A functor F is *seminormal* if it is continuous, monomorphic and preserves intersection, singletons and empty set.

Proposition 2.3.2. *The functors* $\exp, \widetilde{G}, \widetilde{N}_k$, SP^n_G *are normal, and the functors* G, N_k, λ *are weakly normal, the functors* $\exp^c$, $\exp^c_k$ *are almost normal.*

Proof. It is easy to verify that the functor exp satisfies all the conditions from the definition of normal functor. For example, in order to show that exp preserves weight, note that a base of the Vietoris topology is formed by the elements $\langle U_1$, where $U_1, \dots, U_n$ run through a fixed base.

Now we prove that the functor exp is continuous. Let $X = \varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ is an inverse system in **Comp**. The map $h = (\exp p_\alpha)_{\alpha\in\mathcal{A}}$, where $p_\alpha \colon X \to X_\alpha$ are the limit projections, maps $\exp X$ into $\varprojlim \exp \mathcal{S}$.

Suppose that $A, B \in \exp X$ and $A \neq B$, say, $A \setminus B \neq \varnothing$. Let $a \in A$. For every $b \in B$ there exists $\alpha(b) \in \mathcal{A}$ such that $p_{\alpha(b)}(a) \neq p_{\alpha(b)}(b)$. There exists a neighborhood U_b of b such that $p_{\alpha(b)}(a) \notin p_{\alpha(b)}(U_b)$. The open cover $\{U_b \mid b \in B\}$ contains a finite subcover $\{U_{b_1}, \dots, U_{b_k}\}$. Then for every $\alpha \in \mathcal{A}$, $\alpha > \alpha(b_i)$, $i = 1, \dots, k$, we have $\exp p_\alpha(a) \notin \exp p_\alpha(B)$, i. e. $\exp p_\alpha(A) \neq \exp p_\alpha(B)$, and consequently $h(A) \neq h(B)$.

To prove that h is onto, note that, given an element $x = (A_\alpha)_{\alpha\in\mathcal{A}} \in \varprojlim \exp \mathcal{S}$ we have $x = h(A)$, where $A = \cap\{(p_\alpha)^{-1}(A_\alpha) \mid \alpha \in \mathcal{A}\}$.

The almost normality of $\exp^c$ is a consequence of normality of exp. If $f \colon 2^\omega \to [0,1]$ is a surjection, the map $\exp^c f$ is not one (because $[0,1] \notin \exp^c f(\exp^c 2^\omega))$ — this shows that $\exp^c$ is not normal.

The inclusion hyperspace functor fails to preserve preimages. Indeed, consider the map $f \colon \{a,b,c\} \to \{x,y\}$ (all the points under consideration are distinct), $f(a) = f(b) = x$, $f(c) = y$. Let

$$\mathcal{M} = \{\{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\} \in G\{a,b,c\},$$

then $Gf(\mathcal{M}) \in \eta\{x, y\}(x)$, while $\mathcal{M} \notin Gf^{-1}(x) \subset G\{a, b, c\}$. (This example also serves for the functors λ, N and N_k).

□

Proposition 2.3.3. *The probability measure functor P is normal.*

Proof. In order to prove that P preserves weight note that for any dense subset $C \subset C(X)$ the map $\mu \mapsto (\mu(\varphi))_{\varphi \in C}$ is an embedding of the space PX into the product $\prod_{\varphi \in C}[-\|\varphi\|, \|\varphi\|]$ and, by Weierstrass-Stone theorem, one can choose C of cardinality equal to the weight of X.

Now, prove the continuity of P. Suppose that $X = \varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ is an inverse system in **Comp**. The map $h = (Pp_\alpha)_{\alpha \in \mathcal{A}}$, where $p_\alpha \colon X \to X_\alpha$ are the limit projections, maps PX into $\varprojlim P\mathcal{S}$. We have only to show that h is bijective.

Let $\mu_1, \mu_2 \in PX$, $\mu_1 \neq \mu_2$. There exists $\varphi \in C(X)$ such that $\mu_1(\varphi) - \mu_2(\varphi) = a > 0$. By Weierstrass-Stone theorem, the set of functions of the form $\psi \circ p_\alpha$, $\alpha \in \mathcal{A}$, is dense in $C(X)$, thus, there exists $\alpha \in \mathcal{A}$ and $\psi \in C(X_\alpha)$ such that $\|\varphi - \psi \circ p_\alpha\| < a/3$. Since $\|\mu_i\| = 1$, we see that $\|\mu_i(\varphi - \psi \circ p_\alpha)\| < a/3$. Then

$$\begin{aligned}
a = &\|\mu_1(\varphi) - \mu_2(\varphi)\| \\
= &\|\mu_1(\varphi) - \mu_1(\psi \circ p_\alpha) + \mu_1(\psi \circ p_\alpha) - \mu_2(\psi \circ p_\alpha) + \mu_2(\psi \circ p_\alpha) - \mu_2(\varphi)\| \\
\leq &\|\mu_1(\varphi) - \mu_1(\psi \circ p_\alpha)\| + \|\mu_1(\psi \circ p_\alpha) - \mu_2(\psi \circ p_\alpha)\| \\
+ &\|\mu_2(\psi \circ p_\alpha) - \mu_2(\varphi)\| < 2/3a + \|\mu_2(\psi \circ p_\alpha)\|.
\end{aligned}$$

Therefore, $Pp_\alpha(\mu_1)(\psi) = \mu_1(\psi \circ p_\alpha) \neq \mu_2(\psi \circ p_\alpha) = Pp_\alpha(\mu_1)(\psi)$, and $h(\mu_1) \neq h(\mu_2)$. This shows that h is injective.

Now we are going to prove that h is an onto map. Let $\mu = (\mu_\alpha)_{\alpha \in \mathcal{A}} \in \varprojlim P\mathcal{S}$. The equality $\nu_\alpha(\psi \circ p_\alpha) = \mu_\alpha(\psi)$ determines a linear functional ν_α on the subspace $C_\alpha \subset C(X)$ of functions of the form $\psi \circ p_\alpha$, $\psi \in C(X_\alpha)$. We have $C_\alpha \subset C_\beta$ and $\nu_\beta | C_\alpha = \nu_\alpha$, whenever $\alpha < \beta$. Thus, the linear functional $\nu = \cup_{\alpha \in \mathcal{A}} \nu_\alpha$ is determined on the subspace $C = \cup_{\alpha \in \mathcal{A}} C_\alpha$. Clearly, $\|\nu\| = 1$ and ν is nonnegative. By Hahn-Banach theorem, ν can be extended to a nonnegative linear functional $\bar{\nu}$ over $C(X)$ with $\|\bar{\nu}\| = 1$. Obviously, $h(\bar{\nu}) = \mu$.

□

Let $U \colon$ **Conv** $\to$ **Comp** be the forgetful functor. A normal functor F in **Comp** is called *convex* if it can be factored through U, i. e. there

exists a functor $F'\colon \mathbf{Comp} \to \mathbf{Conv}$ such that $F = UF'$. Obviously, P is a convex functor.

Theorem 2.3.4. *A convex functor F is isomorphic to the probability measure functor if and only if it satisfies the condition: for every convex normal functor F' the set of all natural transformations consists of a unique element θ and all the components θX of theta are embeddings. (I. e. P is an initial element of the category of convex normal functors and their natural transformations.)*

Proof. Let F be a convex normal functor. To simplify notation, we use the same notation for the convex normal functors and the corresponding functors into **Conv**. Construct a natural transformation $\theta\colon P \to F$. Since $\eta X(X) \subset F(X)$, we can assign to every $\mu \in PX$ the element $\theta X(\mu) = bX(P\eta X(\mu)) \in \mathrm{conv}(\eta x(X)) \subset FX$ (the barycenter of $P\eta X(\mu) \in PFX$). Obviously, $\theta = (\theta X)$ is a natural transformation.

We are going to prove that ηX is one-to-one, for every X. Consider a finite space X and show that the set $\mathrm{conv}(\eta x(X))$ is an $|X|-1$-dimensional simplex. Indeed, otherwise by the Radon theorem X can be decomposed into two disjoint parts, $X = X_0 \cup X_1$, so that $\mathrm{conv}(X_0) \cap \mathrm{conv}(X_1) \neq \varnothing$. Define the map $g\colon X \to 2$, $g(X_i) = i$, then $Fg\colon FX \to F2$ is an affine map and $Fg(\mathrm{conv}(\eta X(X_i))) = \{\eta 2(i)\}$, $i = 0, 1$, which contradicts to the fact that $\mathrm{conv}(X_0) \cap \mathrm{conv}(X_1) \neq \varnothing$. Thus, for every finite X the set $\mathrm{conv}(\eta X(X))$ is a $|X|-1$-dimensional simplex. Being an affine map, θX is an embedding.

Using continuity of the functors P and F, one can deduce that θX is one-to-one for every zero-dimensional X.

Now, let $g\colon X \to Y$ be an irreducible map of metrizable compacta. Then the map $Fg|\theta X(PX)\colon \theta X(PX) \to \theta Y(PY)$ is also irreducible and, therefore, so is the map $\theta X : PX \to \theta X(PX)$. By the Aleksandrov theorem, the set $\Gamma(X) = \{\mu \in PX \mid \mu = (\theta X)^{-1}\theta X(\mu)$ is a dense G_δ-subset in PX. Note that for every autohomeomorphism $h\colon X \to X$ we have $Ph(\Gamma(X)) = \Gamma(X)$.

A measure $\mu \in PX$ is called *continuous* if it cannot be represented in the form $\alpha\delta_x + (1-\alpha)\mu'$ with $\alpha > 0$.

Lemma 2.3.5. *Suppose that a compact metrizable space X has no isolated point. The set of all continuous measures is a dense G_δ-subset in PX.*

Proof. Let

$$M_n = \{\mu \in PX \mid \mu = \alpha\delta_x + (1-\alpha)\mu' \text{ with } \alpha \geq 1/n\}.$$

We left to the reader verifying that M_n is a closed nowhere dense subset in PX. □

It is also easy to prove that, for every compact metrizable space X without isolated points, the set of all continuous measures μ in PX with $\operatorname{supp}\mu = X$ is a a dense G_δ-subset in PX. Thus, this set intersects $\Gamma(X)$. Let $X = Q$ (the Hilbert cube). We conclude that the set $\Gamma(Q)$ contains a continuous measure μ with $\operatorname{supp}\mu = Q$. By a result from Oxtoby and Prasad [1978], any two such measures μ_1, μ_2 are homeomorphic, i. e. there exists an autohomeomorphism $h\colon Q \times Q$ such that $\mu_2 = Ph(\mu_1)$. This means that all such measures are in $\Gamma(Q)$.

Now we are able to finish the proof of injectivity of θX. Let $\mu, \nu \in PX$.

Case 1. $\operatorname{supp}\mu \neq \operatorname{supp}\nu$, e. g. $\operatorname{supp}\mu \setminus \operatorname{supp}\nu \neq \emptyset$. Choose a map $g\colon X \to I$ such that $g(\operatorname{supp}\nu) = 0$, $g(\operatorname{supp}\mu) \neq 0$. Since $Pg(\nu) = \delta_0 \in \Gamma I$, we see that $\theta I(Pg(\nu)) =\neq \theta I(Pg(\mu))$ and hence $\theta X(\mu) \neq \theta X(\nu)$.

Case 2. $\operatorname{supp}\mu = \operatorname{supp}\nu$. Without loss of generality we may assume that $\operatorname{supp}\mu = X$. Since $\mu \neq \nu$, there exists a closed in X subset B such that $\mu(B) \neq \nu(B)$. Choose a map $g\colon X \to I$ such that $B = g^{-1}(0)$. Then $Pg(\mu) \neq Pg(\nu)$. If $0 \notin \operatorname{Int}_I(g(X))$, there exists a sequence $(t_i)_{i=1}^{\infty}$ in $I \setminus g(X)$ such that $\lim_{i\to\infty} t_i = 0$. Since $Pg(\mu)(\{0\}) \neq Pg(\nu)(\{0\})$, there exists j such that $Pg(\mu)([0,t_j]) \neq Pg(\nu)([0,t_j])$. Define the map $h\colon g(X) \to 2$ by

$$h(t) = \begin{cases} 0, & \text{if } t < t_j, \\ 1, & \text{if } t > t_j. \end{cases}$$

Then $P(h \circ g)(\mu) \neq P(h \circ g)(\nu)$, and injectivity of $\theta 2$ implies that $\theta X(\mu) \neq \theta X(\nu)$.

Now suppose that $0 \in \operatorname{Int}_I(g(X))$. Then $[0, t_0] \subset g(X)$, for some $t_0 > 0$. Define the map $h\colon g(X) \to I$ by the formula

$$h(t) = \begin{cases} t, & \text{if } t \leq t_0, \\ t_0, & \text{if } t \geq t_0. \end{cases}$$

Then $\mu_1 = P(h \circ g)(\mu) \neq P(h \circ g)(\nu) \neq \nu_1$. Let $\lambda \in PQ$ be the Lebesgue measure. As we remarked above, $\mu_2 = \mu_1 \otimes \lambda$, $\nu_2 = \nu_1 \otimes \lambda \in \Gamma([0,t_0] \times Q)$

(note that $[0, t_0] \times Q$ is homeomorphic to Q). But then

$$\theta([0, t_0] \times Q)(\mu_2) = \theta([0, t_0] \times Q)(\nu_2)$$

that gives a contradiction, because $\mu_2 \neq \nu_2$. $\square$

Proposition 2.3.6. *The functor O of order-preserving functionals is weakly normal.*

Proof. The steps of the proof mimick those of Proposition 2.3.3. One has only to use Proposition 2.1.19 instead of the Hahn-Banach theorem.

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be finite compacta (all the point x_1, x_2, x_3, y_1, y_2 are distinct). Define the map $f\colon X \to Y$ as follows: $f(x_1) = y_1$ and $f(x_2) = f(x_3) = y_2$. Consider the functional $\delta_{y_2} \in OY$ supported on $\{y_2\} \subset Y$. Define the functional $\mu \in OX$ by the formula

$$\begin{aligned}\mu(\varphi) = \max\{&\min\{\varphi(x_1), \varphi(x_2)\}, \min\{\varphi(x_1), \varphi(x_3)\},\\ &\min\{\varphi(x_2), \varphi(x_3)\}\}.\end{aligned}$$

It is easy to check that $Of(\mu) = \nu$ and $\mu \notin O\{x_2, x_3\}$. Thus O does not preserve preimages. $\square$

The following result shows that not all the properties from the definition of normal functors are in fact essential.

Theorem 2.3.7. *A functor F:* **Comp** $\to$ **Comp** *is normal iff F is continuous, epimorphic, preserves the limits of the diagrams of the form*

$$A \overset{i}{\hookrightarrow} Y \xleftarrow{f} X,$$

where i is a monomorphism and f is an epimorphism (the last property is formally weaker than the preimage-preserving property).

Proof. The "only if" part is vacuous. To prove the "if" part we first show that F is monomorphic. Let $i\colon A \to B$ be an embedding of discrete spaces and $r\colon B \to A$ be a retraction. Then $F\beta r = F\beta i = \mathrm{id}_{F\beta A}$, so $F\beta i$ is an embedding (recall that β denotes the Stone-Čech compactification functor).

Now, let $j\colon X \to Y$ be an embedding of arbitrary compact Hausdorff spaces. Consider the following commutative diagram in **Top**, which is a universal square:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, "j"] & Y, \end{tikzcd}$$

with discrete A, B and bijective g. Then evidently, the diagram

$$\begin{tikzcd} \beta A \arrow[r, "\beta i"] \arrow[d, "\bar{f}"'] & \beta B \arrow[d, "\bar{g}"] \\ X \arrow[r, "j"] & Y, \end{tikzcd}$$

where the operation $\widetilde{(-)}$ denotes the natural extension, is also an universal square. Since g is a surjection, by the condition of the theorem we obtain that the diagram

$$\begin{tikzcd} F\beta A \arrow[r, "F\beta i"] \arrow[d, "F\bar{f}"'] & F\beta B \arrow[d, "F\bar{g}"] \\ FX \arrow[r, "Fj"] & FY \end{tikzcd}$$

is also a universal square. Using the monomorphity of $F\beta i$ from this we deduce that j is an embedding and the functor F is monomorphic.

No we show that F is intersection-preserving. First consider the case of zero-dimensional compact spaces. Let $D = A \cup B$, $C = A \cap B$, $\dim D = 0$, and D be metrizable. For every map $h\colon D \to D$ such that $h|A = \mathrm{id}_A$ and for every $x \in FA$ we have obviously $Fh(x) = x$.

Let $x \in FA \cap FB$. There exists a retraction $r\colon D \to A$ such that $r(B) = C$. Then $x = Fr(x) \in FC$ and $FC = FA \cap FB$.

Now let D be an arbitrary compact metrizable space and $D = A \cup B$, $C = A \cap B$, A and B being closed subsets of D. There exists a diagram

$$\begin{tikzcd} A' \arrow[d, "f"'] & C' \arrow[l, "i'"'] \arrow[r, "j'"] \arrow[d, "g"] & B' \arrow[d, "h"] \\ A & C \arrow[l, "i"] \arrow[r, "j"'] & B, \end{tikzcd}$$

such that i, j are embeddings, f, g, h are surjections, A', B', C' are zero-dimensional compact metrizable spaces and two squares are universal. Then the maps i', j' are also embeddings and as it is proved earlier we get

$$FA \cap FB = Fg\big((Ff)^{-1}(FA) \cap (Fh)^{-1}(FB)\big) = \\ = Fg(FA' \cap FB') = Fg(FC') = F\big(f(C')\big) = FC.$$

Thus the property of preservation of finite intersection is valid for compact metrizable spaces. The general case (arbitrary intersections of arbitrary compact Hausdorff spaces) can be easily obtained using the continuity of F.

Finally, we show that the preimage-preservation property holds. For a map $f\colon X \to Y$ and $A \subset Y$ closed, let $f = j \circ g$ be a factorization of f such that $g\colon X \to X'$ is a surjection and $j\colon X' \to Y$ is an embedding. Put $A' = j^{-1}(A)$, then two squares in the diagram

$$\begin{tikzcd}
f^{-1}(A) \arrow[r, "g|f^{-1}(A)"] \arrow[d, hook] & A' \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\
X \arrow[r, "g"] & X' \arrow[r, "j"] & Y
\end{tikzcd}$$

are universal. Applying the functor F to this diagram we obtain by the condition of the theorem that the left square transforms into a universal one and by the intersections preserving property the right square also transforms into a universal one. As the result, the outer rectangle transforms into a universal one that exactly is what we require. $\square$

Let $F\colon \mathbf{Comp} \to \mathbf{Comp}$ be an almost normal functor. We define the map $\gamma X\colon \exp X \to \exp FX$ by $\gamma X(A) = FA \subset FX$, $A \in \exp X$.

Theorem 2.3.8. *Let F be an almost normal functor. Then F is epimorphic iff γX is continuous for every X.*

Proof. Let F be epimorphic, $U \subset FX$ open and $FA \in \langle U, FX\rangle$. Then by Proposition 2.2.1 there exist $n \in \mathbb{N}$, a set $\{x_0, \dots, x_{n-1}\}$, a point $a \in FA \cap U$, and a point $b \in Fn$ such that $Ff(b) = a$, where $f\colon n \to X$ is the map sending $i \in n$ to x_i. It follows from Theorem 2.2.3 that there

exist disjoint open neighborhoods $V_0, \dots, V_{n-1}$ of points $x_0, \dots, x_{n-1}$ respectively such that

$$\{Fg(b) \mid g \in C(n, X),\ g(i) \in V_i,\ i \in n\} \subset U.$$

But then we have $\gamma X(B) \in \langle U, FX \rangle$ for each $B \in \langle X, V_0, \dots, V_{n-1} \rangle$ and therefore, the preimage of the set $\langle U, FX \rangle$ under the map γX is open.

Now let $FA \in \langle U \rangle$, i.e., $FA \subset U$. Put

$$\mathcal{B} = \{B \in \exp X \mid A \subset \operatorname{Int} B\}.$$

Then $\bigcap \mathcal{B} = A$ and the intersection preserving property implies that there exist $B_1, \dots, B_k \in \mathcal{B}$ with $FB_1 \cap \dots \cap FB_k \subset U$. Put $B = \bigcap_{i=1}^{n-1} B_i$, then $A \subset \operatorname{Int} B$ and $\gamma X(\langle \operatorname{Int} B \rangle) \subset \langle U \rangle$.

Since the sets of the form $\langle U \rangle$ and $\langle U, FX \rangle$ form an open subbase of the Vietoris topology in $\exp FX$, that completes the proof of continuity of γX.

Conversely, let $f\colon X \to Y$ be a continuous surjective map and C_f be the cylinder of f,

$$C_f = (X \times I) \sqcup Y/\sim,$$

where $(x, 1) \sim f(x)$, $x \in X$. Assume that the sets $X_t = \{(x,t) \mid x \in X\}$. $0 \le t < 1$, and Y are naturally embedded in C_f. Then $\{X_t\}$ tends to Y with $t \to 1$ in $\exp C_f$ and $\{FX_t\}$ tends to FY with $t \to 1$ in $\exp FC_f$. Let $r\colon C_f \to Y$ be the retraction, $r(x,t) = f(x)$, $t < 1$. Then the set $Ff(FX) = Fr\big(\bigcup FX_t \mid t < 1\big)$ is dense in FY and, hence, Ff is surjective. □

Exercise

1. Let (X, ϱ) be a compact metric space. The *continuity metric* ϱ_c is defined as follows: if $A, B \in \exp X$, then $\varrho_c(A, B)$ is the infimum of $\varepsilon > 0$, for which there exist maps $\varphi\colon A \to B$, $\psi\colon B \to A$ such that

$$\varrho(x, \varphi(x)) \le \varepsilon, \quad \varrho(y, \psi(y)) \le \varepsilon$$

for every $x \in A$, $y \in B$. (Note that the continuity metric needs *not* be a continuous metric on $\exp X$. Find an appropriate formulation and prove the continuity of the normal functors in the metric ϱ_c.

Problem

1. Is every almost normal functor a subfunctor of a normal functor?

2.4. Supports

Suppose that F is a monomorphic functor that preserves the intersections, and $a \in FX$. The *support* of a is the set

$$\mathrm{supp}_{F,X}(a) = \bigcap\{A \subset X \mid A \text{ is closed and } a \in FA\}.$$

In the obvious situations the notation $\mathrm{supp}_{F,X}(a)$ is abbreviated to $\mathrm{supp}_F(a)$ or even to $\mathrm{supp}(a)$.

Remark 2.4.1. The preimage-preserving property can be also defined by means of the notion of support. Namely, a functor F is said to *preserve supports* if $\mathrm{supp}\, Ff(a) = f\big(\mathrm{supp}(a)\big)$ for every $f\colon X \to Y$ and $a \in X$. Obviously, a functor preserves preimages iff it preserves supports.

Proposition 2.4.2. *The map* $\mathrm{supp}_{F,X}\colon FX \to \exp X$ *is lower semicontinuous.*

Proof. Given an open subset U in X we see that the set

$$\{a \in FX \mid \mathrm{supp}_{F,X}(a) \cap U \neq \varnothing\} = FX \setminus F(\mathrm{supp}_{F,X}(a))$$

is open, because the set $F\big(\mathrm{supp}_{F,X}(a)\big)$, being compact, is closed in FX. □

Definition 2.4.3. A functor F is called a *functor with continuous supports* if the map $\mathrm{supp}_F\colon FX \to \exp X$ is continuous for every X.

Theorem 2.4.4. *Let F be a normal functor. The map* $\mathrm{supp}_F\colon FX \to \exp X$ *is continuous for all X whenever it is continuous for finite X.*

Proof. Show first the continuity of supp_F for zero-dimensional compact spaces. Let $\dim Z = 0$. Then there exists an inverse system $\mathcal{S} = \{Z_\alpha, p_{\alpha\beta}\}$ of finite spaces, for which $Z = \varprojlim \mathcal{S}$. Therefore, the family $\{\mathrm{supp}_F \mid FZ_\alpha \to \exp Z_\alpha\}$ is a morphism of the inverse system $F(\mathcal{S})$ to the inverse system $\exp \mathcal{S}$. The map $\mathrm{supp}_F\colon FZ \to \exp Z$ is its limit one, and thus, it is continuous.

Now let X be an arbitrary compact Hausdorff space. There exists a surjective map $g\colon Z \to X$ of a zero-dimensional compact space Z. By commutativity of the following diagram

$$\begin{array}{ccc} FZ & \xrightarrow{Fg} & FX \\ {\scriptstyle \mathrm{supp}_F}\downarrow & & \downarrow{\scriptstyle \mathrm{supp}_F} \\ \exp Z & \xrightarrow{\exp g} & \exp X, \end{array}$$

closeness of Fg, and the previous paragraph, the map $\operatorname{supp}_F\colon FX \to \exp X$ is continuous. □

Definition 2.4.5. A functor F is said to be *finite* if it preserves the class of finite discrete spaces.

Corollary 2.4.6. *Any finite normal functor is a functor with continuous supports.*

If a functor F has continuous supports, then, evidently, the family

$$\{\operatorname{supp}_F\colon FX \to \exp X \mid X \in \mathbf{Comp}\}$$

forms a natural transformation $\operatorname{supp}_F\colon F \to \exp$.

Proposition 2.4.7. *Let F be a weakly normal functor with continuous supports. Then F is normal.*

Proof. Suppose that F fails to preserve preimages, i.e. there exist a map $f\colon X \to Y$, a closed subset $A \subset Y$, and $a \in FX$ such that $Ff(a) \in FA$ and $\operatorname{supp}(a) \not\subset f^{-1}(A)$. Imbed the map f into the retraction $r\colon Y \times I^\tau \to Y \times \{0\}$, $r(y,x) = (y,0)$, for some τ. For each $t \in I$ let $r_t\colon Y \times I^\tau \to Y \times I^\tau$ acts by $r_t(y,x) = (y,tx)$. Then $\lim_{t\to 0} Fr_t(a) = Ff(a)$ but $\lim_{t\to 0} \operatorname{supp} Fr_t(a) \not\subset A$ thus contradicting to continuity of supports. □

Proposition 2.4.8. *Let $F\colon \mathbf{Comp} \to \mathbf{Comp}$ be a monomorphic functor that preserves intersection, singletons, and empty set. Then the family of maps $\eta X\colon X \to FX$ determines the natural transformation $\eta\colon \mathrm{Id} \to F$.*

Proof. Since the map $\operatorname{supp}\colon FX \to \exp X$ is lower semicontinuous, the set $\eta X(X) = \{a \in FX \mid |\operatorname{supp}(a)| = 1\}$ is a closed subset of X. Moreover, for every closed subset A in X the set $\eta X(A) = \eta X(X) \cap FA$ is a closed subset of $\eta X(X) \subset FX$. Thus, the map ηX is a closed map into a compact Hausdorff space, and therefore an embedding. □

Remark 2.4.9. In Proposition 2.4.8, one cannot weaken the condition of preserving intersection to one of preserving finite intersections. Indeed, consider the functor $\beta_{\mathrm{d}}\colon \mathbf{Comp} \to \mathbf{Comp}$, where $\beta_{\mathrm{d}}X$ is the Stone-Čech compactification of the discrete copy X_{d} of the space X. It is easy to see that the map $\eta X\colon X \to \beta_{\mathrm{d}}X$ is discontinuous for any infinite X.

Proposition 2.4.10. *For a weakly normal functors F', F'' and $a \in F''F'X$ the set* $\mathrm{supp}_{F''F'}(a)$ *is the closure of the set* $\cup\{\mathrm{supp}_{F'}(b) \mid b \in \mathrm{supp}_{F''}(a)\}$.

Proof. The inclusion $\subset$ is obvious. Let $a \in F''F'B$, where $B \in \exp X$. Then $\mathrm{supp}_{F''}(a) \subset F''B$, i. e. $b \in F'B$ for every $b \in \mathrm{supp}_{F''}(a)$, hence $\mathrm{supp}_{F'}(b) \subset B$. Therefore, $\cup\{\mathrm{supp}_{F'}(b) \mid b \in \mathrm{supp}_{F''}(a)\} \subset B$ and we obtain the inclusion $\supset$. □

Proposition 2.4.11. *Let $F = \prod\{F_\alpha \mid \alpha \in \mathcal{A}\}$, $a = (a_\alpha) \in FX$, where $a_\alpha \in F_\alpha X$. Then* $\mathrm{supp}_F(a)$ *is the closure of the set* $\cup\{\mathrm{supp}_{F_\alpha}(a_\alpha) \mid \alpha \in \mathcal{A}\}$.

2.5. Functors of finite degree

Let F be a monomorphic functor that preserves intersections. The *degree of a point $a \in FX$* is the cardinality of $\mathrm{supp}(a)$, whenever $\mathrm{supp}(a)$ is finite and ∞, otherwise. The degree of a is denoted by $\deg(a)$. The maximal possible value of $\deg(a)$ is called the *degree of the functor* F and is denoted by $\deg(F)$. If F is not a functor of degree n for any $n \in \mathbb{N}$ then is said to be a functor of *infinite degree* (this is denoted by $\deg F = \infty$).

The following are immediate consequences of Propositions 2.4.10 and 2.4.11.

Proposition 2.5.1. *Let F', F'' be (weakly, almost) normal functors. Then* $\deg(F''F') = \deg(F'')\deg(F')$.

Proposition 2.5.2. *Let $F = \prod\{F_\alpha \mid \alpha \in \mathcal{A}\}$, then*

$$\deg(F) = \sum\{\deg(F_\alpha) \mid \alpha \in \mathcal{A}\}$$

(here $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ is a family of (weakly, almost) normal functors.

Proposition 2.5.3. *Let F be a weakly normal functor. If it has continuous supports and* $\deg F = \infty$ *then the map* $\mathrm{supp}_F\colon FX \to \exp X$ *is surjective for every X.*

Proof. Show first that the statement of proposition holds for finite Hausdorff spaces. To the contrary, let A be a finite space with $\mathrm{supp}_F(a) \neq A$ for all $a \in FA$. Since $\deg F = \infty$, there exist a zero-dimensional compact Hausdorff space Z, $|Z| > |A|$, and $b \in FZ$ such that $\mathrm{supp}_F(b) = Z$.

Let $r\colon Z \to A$ be a surjective map. We obtain that $\operatorname{supp}_F(Fr(b)) = r(Z) = A$, a contradiction.

Finally, since the set $\{A \in \exp X \mid |A| < \omega\}$ is dense in $\exp X$, $X \in \mathbf{Comp}$, one can prove the surjectivity of $\operatorname{supp}_F\colon FX \to \exp X$, using its closedness. □

Theorem 2.5.4. *Let F be a normal functor and $|Fk| \leq 2^k - 1$, $k \in \mathbb{N}$. Then $F \cong \exp$ whenever $\deg(F) = \infty$, and $F \cong \exp_n$ whenever $\deg(F) = n$.*

Proof. At first, let $\deg F = \infty$. Since F is finite, by Corollary 2.4.6 it has continuous supports. Thus, by Proposition 2.5.3 the map $\operatorname{supp}_F\colon FX \to \exp X$ is surjective for all X. Thus, it is one-to-one for finite X.

Representing any zero-dimensional compact Hausdorff space Z as the limit space of an inverse system of finite discrete spaces, one can verify that $\operatorname{supp}_F\colon FZ \to \exp Z$ is a bijection. Hence, this map is a homeomorphism for every zero-dimensional Z.

Since every compact Hausdorff space X is an image of some zero-dimensional compact Hausdorff space Z at an irreducible map $Z \to X$, by epimorphness of F we have the injectivity of $\operatorname{supp}_F\colon FX \to \exp X$.

Hence, $\operatorname{supp}_F\colon F \to \exp$ is a functorial isomorphism.

Now let $\deg F = n$, $n < \infty$. □

Examples. There exists a normal functor F with finite supports which is not a functor of finite degree.

Let F be a subfunctor of the probability measure functor,

$$FX = \left\{\sum_{i=1}^{k} \alpha_i \delta_{x_i} \mid k \in \mathbb{N} \text{ and there exists } j \text{ such that } \alpha_j \geq (1 - 1/k)\right\}.$$

The details are left to the reader.

2.5.1. Subfunctors of free topological group functor

A wide class of examples of functors of finite degree can be found in topological algebra. Let $X \in |\mathbf{Comp}|$. A *free topological (Abelian) group* of X is a topological group $F(X)$ $(A(X))$ satisfying the properties:

1) X is a subset of $F(X)$ $(A(X))$;

2) every continuous map of X into (the underlying space of) a topological (Abelian) group G can be continuously extended over X.

It is well-known that the free topological groups exist and are determined uniquely up to topological isomorphism. Moreover, it can be easily deduced from the definition of $F(X)$ that the subspace X algebraically generates $F(X)$. Denote by $\tilde{X}$ the subspace $X \cup X^{-1} \cup \{e\}$ of $F(X)$ (the operations in $F(X)$ are denoted multiplicatively and $\{e\}$ is the neutral element) and, for every $n \in \mathbb{N}$, let $F_n(X) = \tilde{X} \dots \tilde{X}$ (n factors).

Suppose that $f \colon X \to Y$ is a morphism in **Comp**. Then the map $f \colon X \to Y \subset F_n(Y)$ can be uniquely extended to a map $F(f) \colon F(X) \to F(Y)$ and it is easy to see that $F(f)(F_n(X)) \subset F_n(Y)$. Denoting by $F_n(f)$ the restriction of the map $F(f)$ onto $F_n(X)$ (considered as a map into $F_n(Y)$) we obtain a covariant functor F_n. It is easy to see that F_n is monomorphic and preserves intersections. Thus, a notion of support is defined for F_n. Obviously, $\deg F_n = n$.

The topology of the space $F_n(X)$ is exactly the quotient topology generated by the multiplication map $\tilde{X}^n \to F(X)$, $(x_1, \dots, x_n) \mapsto x_1 \dots x_n$.

Similarly, the functors A_n can be defined.

2.5.2. Functors generated by Hartman-Mycielski construction

Here we consider the functors of finite degree related to the Hartman-Mycielski construction. Let $X \in |\mathbf{Comp}|$, $n \in \mathbb{N}$. Denote by HM_nX the set of (not necessarily continuous) maps $\alpha \colon [0,1) \to X$ satisfying the condition:

there exist $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1 \in [0,1]$, $1 \leq k \leq n$, such that α is constant on each $[t_i, t_{i+1})$.

For every entourage U of the diagonal Δ_X and $\varepsilon > 0$, let

$$\langle \alpha, U, \varepsilon \rangle = \{\alpha' \in HM_nX \mid m\{t \in [0,1) \mid (\alpha(t), \alpha'(t)) \notin U\} < \varepsilon\}$$

(here m is the Lebesgue measure on $[0,1)$). It is easy to show that the sets $\langle \alpha, U, \varepsilon \rangle$ form a base of a topology in HM_nX. Given a map $f \colon X \to Y$ in **Comp**, define a map $HM_nf \colon HM_nX \to HM_nY$ by the formula: $HM_nf(\alpha) = f \circ \alpha$. Obviously, HM_n is a functor of degree n in **Comp**.

2.5.3. Projective power functors

Let $X \in |\mathbf{Comp}|$, $X \neq \varnothing$ and $n \in \mathbb{N}$. A point $y \in X$ is called *essential* for $(x_0, \ldots, x_{n-1}) \in X^n$, whenever the number $|\{j \mid y = x_j\}$ is odd. Denote by $\mathcal{R}$ the following equivalence relation on X^n:

$$x = (x_0, \ldots, x_{n-1})\mathcal{R}y = (y_0, \ldots, y_{n-1})$$
$$\iff \text{the sets of essential coordinates of } x \text{ and } y \text{ coincide.}$$

The quotient space $\operatorname{Pr}^n X = X^n/\mathcal{R}$ is called the *n-th projective power* of X. The equivalence class of $\mathcal{R}$ containing the element $(x_0, \ldots, x_{n-1}) \in X^n$ will be denoted by $\langle x_0, \ldots, x_{n-1}\rangle$. For any morphism $f\colon X \to Y$ in $|\mathbf{Comp}|$ define the map $\operatorname{Pr}^n f\colon \operatorname{Pr}^n X \to \operatorname{Pr}^n Y$ by the formula

$$\operatorname{Pr}^n f\langle x_0, \ldots, x_{n-1}\rangle = \langle f(x_0), \ldots, f(x_{n-1})\rangle.$$

It is convenient to define $\operatorname{Pr}^n \varnothing = \varnothing$.

Proposition 2.5.5. *For n odd, the functor Pr^n is weakly normal.*

Note that for even n the functors Pr^n fail to preserve intersections. Indeed, if $x, y \in X$, $x \neq y$, then $\operatorname{Pr}^n\{x\} = \operatorname{Pr}^n\{y\}$ and therefore

$$\operatorname{Pr}^n\varnothing = \operatorname{Pr}^n(\{x\} \cap \{y\}) \neq \operatorname{Pr}^n(\{x\}) \cap (\operatorname{Pr}^n\{y\}).$$

Exercises

1. Let F be a normal functor, $\deg(F) = 2$ and $FS^1 = S^3$. Show that F is isomorphic to P_2.
2. Show that the space $\exp_3 S^1$ is homeomorphic to S^3.
3. Verify that normality of F is essential in Theorem 2.5.4.
4. For $X \in |\mathbf{Comp}|$ define a map $\xi X\colon HM_nX \to P_nX$ by the following manner. If $\alpha \in HM_nX$ and there exist $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1 \in [0,1]$, $1 \leq k \leq n$, such that α is constant (and its value is x_i) on each $[t_i, t_{i+1})$, then let $\xi X(\alpha) = \sum_{i=0}^{n-1}(t_{i+1} - t_i)\delta_{x_i}$. Prove that $\xi = (\xi X)_{X \in |\mathbf{Comp}|}$ is a natural transformation of HM_n to P_n.
5. Let F be an almost normal functor. Show that F is epimorphic if and only if, for every $X \in |\mathbf{Comp}|$, the set $F_\omega X$ is dense in FX.
6. Let $FX = \{\mathcal{A} \in \exp^2 X \mid \mathcal{A}$ is contained in a unique maximal linked system $\}$. Prove that F is a normal subfunctor of $\exp^2$.

Problems

1. Is there a normal functor F such that $F_n = HM_n$?
2. Describe all weakly normal $\perp$-invariant subfunctors of the functor G. What is the minimal $\perp$-invariant subfunctor F of the functor G that contains N?

2.6. The category of normal functors

The following proposition allow to consider the category of (weakly, almost) normal functors with their natural transformations. Denote this category by **NF** (**WNF**, **ANF**).

Proposition 2.6.1. *Let $\varphi_1, \varphi_2 \colon F \to F'$ be natural transformations of (weakly, almost) normal functors. Then $\varphi_1 = \varphi_2$ iff $\varphi_1 Q = \varphi_2 Q$.*

Proof. We have to show only the sufficient part. For every τ consider an inverse system $\mathcal{S} = \{Q^A, \mathrm{pr}^A_B; \mathcal{P}_\omega(\tau)\}$, where $\mathcal{P}_\omega(\tau)$ is the set of countable subsets of τ, ordered by the inclusion relation, and $\mathrm{pr}^A_B \colon Q^A \to Q^B$, $B \subset A$, is the projection. Then $\varprojlim \mathcal{S} = (Q^\tau, \mathrm{pr}_A)$ (here $\mathrm{pr}_A \colon Q^\tau \to Q^A$, $B \subset A$, is a projection). Since for every $A, B \in \mathcal{P}_\omega(\tau)$, $B \subset A$, the following diagram

$$\begin{tikzcd}
F(Q^A) \arrow[r, "\varphi_i Q^A"] \arrow[d, "F\,\mathrm{pr}^A_B"'] & F'(Q^A) \arrow[d, "F'\,\mathrm{pr}^A_B"] \\
F(Q^B) \arrow[r, "\varphi_i Q^B"] & F'(Q^B)
\end{tikzcd}$$

is commutative and the functors F, F' are continuous, we have $\varphi_1 Q^\tau = \varprojlim\{\varphi_1 Q^A\} = \varprojlim\{\varphi_2 Q^A\} = \varphi_2 Q^\tau$.

Now let X be an arbitrary compact Hausdorff space and $i \colon X \to Q^\tau$ an embedding for some τ. Then

$$F'i \circ \varphi_1 X = \varphi_1 Q^\tau \circ Fi = \varphi_2 Q^\tau \circ Fi = F'i \circ \varphi_2 X.$$

Since $F'i$ is an embedding, $\varphi_1 X = \varphi_2 X$. □

Proposition 2.6.2. *Let $\varphi \colon F \to F'$ be a natural transformation of weakly normal functors. If the component φQ of φ is a homeomorphism then φ is a functorial isomorphism.*

Proof. Let X be a compact metrizable space and $X \to Q$ an embedding. Since $\varphi Q \circ F_i = F'i \circ \varphi X$, the map φX is an embedding too. Consider a surjection $f \colon A \to Q$ of a zero-dimensional compact metrizable space A onto Q. Fix $a \in F'X$ and choose $b \in FQ$ and $c \in FA$ such that $\varphi Q(b) =$

a, $Ff(c) = b$. Let $r\colon \mathrm{supp}_F(c) \to \mathrm{supp}_{F'}(\varphi A(c))$ be a retraction and $d = F(f \circ r)(c)$. Then

$$\begin{aligned}\varphi Q(d) =& \varphi Q \circ F(f \circ r)(c) = F'(f \circ r) \circ \varphi A(c)\\ =& F'f \circ \varphi A(c) = \varphi Q \circ Ff(c) = a\end{aligned}$$

and

$$\mathrm{supp}_F(d) = f\big(\mathrm{supp}_{F'}(\varphi A(c))\big) \subset f(f^{-1}X) = X,$$

i.e., $d \in FX$. Thus the map φX is a homeomorphism. The continuity of F implies that φ_X is a homeomorphism for every compact Hausdorff space X. □

Theorem 2.6.3. *Every morphism of the category* **NF** *has an epi-mono-factorization.*

Proof. Let $\varphi\colon F \to F'$ be a natural transformation of normal functors. Define a subfunctor F'' of F' by the formula $F''X = \varphi X(FX)$, $X \in$ **Comp**. Suppose that F'' is a normal functor. Then for a natural embedding $i\colon F'' \to F'$ and a natural surjection $\varphi'\colon F \to F''$ we obtain a epi-mono-factorization $\varphi = i \circ \varphi'$ of φ.

Hence, we have only to show normality of F''. At first prove two claims.

Claim 2.6.4. *Let $X = A \cup B$ and $A \cap B = C$ for closed subsets A, B of X. If there exists a homeomorphism $h\colon A \to B$ with $h|C = 1_C$, then $F''A \cap F'C = F''B \cap F'C$.*

Proof. Consider $a \in F''A \cap F'C$ and $b \in FA$ such that $a = \varphi A(b)$. Since $A \in F'C$, we have

$$\varphi B \circ Fh(b) = F'h \circ \varphi A(b) = F'h(a) = a.$$

Hence, $a \in F'' \cap F'C$. □

Claim 2.6.5. *Let C be a closed subset of A. Then $F''A \cap F'C = F''C$.*

Proof. At first, suppose that C is a retract of A. Identifying A with some subspace of the Tychonov cube I^τ, one can construct sequences of closed subsets $\{A_i \mid i < \omega\}$ of I^τ and homeomorphisms $h_i\colon A \to A_i$ satisfying the following:

1) $A_i \supset C$, $h_i|C = \mathrm{id}$, $i < \omega$;

2) the sequence $\{h_i \mid i < \omega\}$ uniformly tends to a retraction $r\colon A \to C$.

By Claim 2.6.4, a map $F'h_i|(F''A \cap F'C)$ is identical for every $i < \omega$. Hence,

$$F''A \cap F'C \supset F''r(F''A) \cap F'r(F'C),$$

and this implies the claim statement.

Generally, consider a surjection $f\colon A' \to A'$ such that a set $C' = f^{-1}(C)$ is a retract of A'. Let $a \in F''A \cap F'C$ and $b \in F''A'$ satisfy the following equality $F''f(b) = a$. By the preimage preserving property of F' obtain that

$$b \in F'f^{-1}(F'C) = F'(f^{-1}C) = F'C'.$$

Then $b \in F''A' \cap F'C'$ and by the previous claim $b \in F''C'$. Hence, $a \in F''C$, and we proved the claim. □

Continue the proof of the theorem. Show that the functor F'' preserves the limit of the following diagram

$$A \hookrightarrow Y \xleftarrow{f} X.$$

Indeed, if $a \in F''f^{-1}(F''A)$, then

$$a \in F''X \cap F'f^{-1}(F'A),$$

and by Lemma 2.6.5, we have $a \in F''(f^{-1}A)$. Obviously, F'' satisfies also the other conditions of Theorem 2.3.7. Remark only, that the continuity of F'' follows from Proposition 2.2.2. Hence, F'' is a normal functor, and this finishes the proof. □

We will use the following construction due to V. Basmanov. For a functor F define the map $\pi_{F,n}X\colon X^n \times Fn \to FX$ by the formula $\pi_{F,n}X(f,x) = Ff(x)$. The map $\pi_{F,n}X$ is called the *Basmanov map*.

Proposition 2.6.6. *The maps $\pi_{F,n}X$ are continuous and form the natural transformation $\pi_n\colon (-)^n \times Fn \to F$.*

Proof. Offering to the reader to verify the naturality of π_n, remark only that the continuity of these maps is obvious by Theorem 2.2.3. □

Denote by $\mathbf{K}_n$, $n \geq 1$, a full subcategory of the category **Comp** whose objects are the discrete spaces of cardinality $\leq n$.

Proposition 2.6.7. *Let $F, F'\colon \mathbf{Comp} \to \mathbf{Comp}$ be weakly normal functors with $\deg F \le n$. For each natural transformation $\varphi\colon F|\mathrm{K}_n \to F'|\mathrm{K}_n$ there exists a unique natural transformation $\bar\varphi\colon F \to F'$ extending φ. If φ is a functorial isomorphism and $\deg F' \le n$ then $\bar\varphi$ is also functorial isomorphism.*

Proof. Uniqueness. Let $x \in FX$. Since $\deg(F) \le n$, there exist a map $f\colon n \to X$ and a point $y \in Fn$ with $x = Ff(y)$. Then

$$\bar\varphi X(x) = \bar\varphi X \circ Ff(y) = F'f \circ \varphi n(y).$$

Existence. If $x \in FX$ and $x = Ff(y)$ for some $f\colon n \to X$ and $y \in Fn$, we set $\bar\psi(x) = F'f \circ \psi n(y)$. To verify that this construction is well defined, consider $y' \in Fn$ and $f'\colon n \to X$ with $x = Ff'(y')$. Let

$$n \xrightarrow{g} i \xrightarrow{h} X, \qquad n \xrightarrow{g'} i' \xrightarrow{h'} X$$

be epi-mono factorizations of f, f' respectively. Let also the following diagram

$$i \xleftarrow{k} j \xrightarrow{k'} i'$$

be the limit of $i \xrightarrow{h} X \xleftarrow{h'} i'$.

By the intersection preserving property of F there exists $z \in Fj$ such that $Fk(z) = Fg(y)$, $Fk'(z) = Fg'(y')$. Then

$$\begin{aligned} Ff' \circ \psi n(y') &= F'(h' \circ g') \circ \psi n(y') = F'h' \circ \psi i' \circ Fg'(y) = \\ &= F'h' \circ \psi i' \circ Fk'(z) = F'h' \circ F'k' \circ \psi j(z) = \\ &= F'h \circ F'k \circ \psi j(z) = F'h \circ \psi j \circ Fk(z) = \\ &= F'h \circ \psi i \circ Fg(y) = F'h \circ F'g \circ \psi n(y) = F'f \circ \psi n(y). \end{aligned}$$

Show the commutativity of the following diagram

$$\begin{array}{ccc} FX & \xrightarrow{\bar\psi X} & F'X \\ {\scriptstyle Fs}\downarrow & & \downarrow{\scriptstyle F's} \\ FY & \xrightarrow{\bar\psi Y} & F'Y \end{array}$$

for any continuous map $s\colon X \to Y$. Let $x \in FX$, and $x = Ff(y)$ for some $f\colon n \to Y$ and $y \in Fn$. Then we obtain

$$F's \circ \bar{\psi}X(x) = F's \circ F'f \circ \psi n(y) = \bar{\psi}Y \circ F(s \circ f)(y) = \bar{\psi}Y \circ Fs(x).$$

To verify continuity of $\bar{\psi}X$ consider the following diagram:

$$\begin{tikzcd} Fn \times C(n,X) \arrow[r, "\pi_{F,n}X"] \arrow[d, "\psi n \times \mathrm{id}"'] & FX \arrow[d, "\bar{\psi}X"] \\ F'n \times C(n,X) \arrow[r, "\pi_{F',n}X"] & F'X, \end{tikzcd}$$

where $\pi_{F,n}X, \pi_{F',n}X$ are the Basmanov maps. This diagram is commutative, because

$$\begin{aligned} \bar{\psi}X \circ \pi_{F,n}X(x,f) = \bar{\psi}X \circ Ff(x) = F'f \circ \psi n(x) = \\ = \pi_{F',n}X\big(\psi n(x), f\big) = \pi_{F',n}X(\psi n \times \mathrm{id})(x,f). \end{aligned}$$

Since the maps $\pi_{F,n}X$ are quotient, we have continuity of $\bar{\psi}X$.

Finally, if $\deg F' \le n$ and $\psi\colon F|\mathbf{K}_n \to F'|\mathbf{K}_n$ is a functorial isomorphism, then there exists a natural transformation $\chi\colon F'|\mathbf{K}_n \to F|\mathbf{K}_n$ such that $\psi \circ \chi = \mathrm{id}_{F'|\mathbf{K}_n}$, $\chi \circ \psi = \mathrm{id}_{F|\mathbf{K}_n}$. Therefore, $\bar{\psi} \circ \bar{\chi} = \mathrm{id}_{F'}$, $\bar{\chi} \circ \bar{\psi} = \mathrm{id}_F$, and $\bar{\psi}$ is a functorial isomorphism. □

By this theorem we immediately obtain a stronger version of Proposition 2.2.7.

Corollary 2.6.8. *For every weakly normal functor F in the category* **Comp** *there exists a unique natural transformation $\eta\colon \mathrm{Id} \to F$.*

Remark 2.6.9. Both Proposition 2.2.7 and Corollary 2.6.8 use the continuity of functors. In fact, for the existence (and uniqueness) of a natural transformation $\mathrm{Id} \to F$ one only needs for the functor F to be monomorphic and preserve empty set, singletons, and intersections (see E. Shchepin [1981]). The reader can prove this as an exercise.

Proposition 2.6.10. *The category* **NF** *(***WNF**, **ANF***) has finite limits and countable colimits.*

Proof. The products are counted componentwisely:

$$\left(\prod\{F_\alpha \mid \alpha \in A\}\right)X = \prod\{F_\alpha X \mid \alpha \in A\}.$$

The coproduct $F = F' \sqcup F''$ of functors is defined by the formula

$$FX = (F'X \sqcup F''X)/\mathcal{R}_X,$$

where all the non-trivial classes of the equivalence relation $\mathcal{R}_X$ are of the form $\{\eta' X(x), \eta'' X(x)\}$, $x \in X$, with obvious action on morphisms (here $\eta'\colon \mathrm{Id} \to F'$, $\eta''\colon \mathrm{Id} \to F''$ are determined by Corollary 2.6.8). □

Example. Epi-mono-factorization in **NF** is not unique. Indeed, define the functor $e\colon \mathbf{Comp} \to \mathbf{Comp}$ as follows. For each X let $\mathcal{R}_X$ denote the equivalence relation on the space $\exp X \times 2$: $(A, 0)\mathcal{R}_X(A, 1)$ iff $A \in \exp^c X$. Put $eX = (\exp X \times 2)/R_X$. Thus, we evidently obtain the quotient-functor e of $\exp(-) \times 2$. It is easily to check that e is normal. The quotient natural transformation $q\colon \exp(-) \times 2 \to e$ factors through the natural transformation $\varphi\colon \exp \sqcup \exp \to e$, and the restriction of the map φX onto $(\exp \sqcup \exp)_\omega X$ is an embedding. Hence, φ is a monomorphism in **NF**. Since φ is evidently an epimorphism, we obtain that the category **NF** contains the bimorphism φ that is not an isomorphism. It can be easily deduced from this that epi-mono-factorization is not unique in **NF**.

Moreover, the restrictions of functors e and $\exp \sqcup \exp$ onto the subcategory $\mathbf{Comp}_0$ of zero-dimensional compact Hausdorff spaces are isomorphic. Hence, in general it is impossible to extend a natural transformation given on $\mathbf{Comp}_0$ onto **Comp**.

Proposition 2.6.11. *A normal functor F is isomorphic to* exp *iff*

$$F(X \sqcup Y) = (FX) \sqcup (FY) \sqcup (FX \times FY)$$

for every X and Y.

Proof. Induction by n. Assume that for each $k < n$ it is shown that $|Fk| \le 2^{k-1}$. Then

$$|Fn| = |F((n-1) \sqcup 1)| = |F(n-1) \sqcup (F1) \sqcup \left(F(n-1) \times F1\right)| =$$
$$= 2|F(n-1)| + 1 \le 2(2^{n-1} - 1) + 1 = 2^n - 1$$

and by Theorem 2.5.4, we have $F \cong \exp$. □

We define the $(n+1)$*-dimensional simplex functor* $\Delta^{n+1}\colon \mathbf{K}_n \to \mathbf{Comp}$ by the following manner. Let X be an object of $\mathbf{K}_n$. Then, by definition, $\Delta^{n+1}X$ is a $(|X|+1)$-dimensional simplex whose vertices are identified with the points of X. A morphism $f\colon X \to Y$ in $\mathbf{K}_n$ uniquely determines an affine map $\Delta^{n+1}f\colon \Delta^{n+1}X \to \Delta^{n+1}Y$. It is easy to check that Δ^{n+1} is a normal functor.

Theorem 2.6.12. *Let $F\colon \mathbf{Comp} \to \mathbf{Comp}$ be a normal functor such that $F|\mathbf{K}_n\colon \Delta^{n+1}$ for each $n \in X$. Then $F \cong P$.*

Proof. It can be easily deduced from continuity of the functors F and P that $F|\mathbf{Comp}_0 \cong P|\mathbf{Comp}_0$. For the sake of simplicity we assume that $F|\mathbf{Comp}_0 = P|\mathbf{Comp}_0$.

Let X be an arbitrary compact Hausdorff space and $f\colon X' \to X$ be a Milutin map, where X' is zero-dimensional compact Hausdorff space (see Proposition 1.1.9). Then there exists an affine map $s\colon PX \to PX'$ such that $Pf \circ s = 1_{PX}$. Define the map $\varphi X\colon PX \to FX$ as the composition

$$PX \xrightarrow{s} PX' = FX' \xrightarrow{Ff} FX.$$

In order to verify that φX is well defined, remark that φX is uniquely determined on the dense subset $\{\mu \in PX \mid |\mathrm{supp}(\mu)| < \omega\}$, by normality of P and F. It is easy to show that $\varphi = (\varphi X)$ is a natural transformation extending the identity transformation of the restrictions of the functors P and F onto $\mathbf{Comp}_0$.

Show that for each compact Hausdorff space X the map φX is injective. Assuming the contrary, we obtain that there exist $\mu_1, \mu_2 \in PX$, $\mu_1 \neq \mu_2$, for which $\varphi X(\mu_1) = \varphi X(\mu_2)$. It is not difficult to construct the map $f\colon X \to I = [0,1]$ with $Pf(\mu_1) \neq Pf(\mu_2)$. Let $A \subset I$ be a closed subset such that $\mu_1(A) \neq \mu_2(A)$ and $g\colon I \to I$ a map with $g^{-1}(1) = A$. Then

$$Pg \circ Pf(\mu_1) \neq Pg \circ Pf(\mu_2).$$

Consider the sequence of maps $h_i\colon I \to I$, $i \in \mathbb{N}$, defined by the formula $h_i(t) = t^i$, $t \in I$, $i \in \mathbb{N}$.

If ν_i is the limit point of the sequence $(Ph_i \circ Pg \circ Pf(\mu_j))_{i=1}^{\infty}$, $j = 1, 2$, then evidently $\mathrm{supp}(\nu_j) \subset \{0,1\}$, $\nu_1 \neq \nu_2$ but $\varphi I(\nu_1) = \varphi I(\nu_2)$, by zero-dimensionality of $\mathrm{supp}(\nu_1)$ and $\mathrm{supp}(\nu_2)$, and we get a contradiction.

It is obvious that φX is surjective. □

As an application of Proposition 2.6.7 we characterize the functor P_2.

Proposition 2.6.13. *A normal functor F in $\mathbf{Comp}$ is isomorphic to P_2 iff $\deg F = 2$ and $(F2, F_1 2) \cong (I, \{0,1\})$.*

Proof. The "only if" part is obvious. To prove the "if" part let $h: 2 \to 2$ be the non-trivial involution and a be the fixed point of the map Fh. Suppose $F\{0\} = \{0\}$, $F2 = I$, and fix a homeomorphism $g': [0, a] \to \{\mu \in P_2 2 \mid \mu(\{0\}) \leq \frac{1}{2}\}$. Then we can extend g' to the homeomorphism $g2: [0, 1] \to P_2 2$ such that $P_2 h \circ g2 = g2 \circ Fh$. The map $g2$ can be obviously supplemented to the natural transformation $g: F|\mathbf{K}_2 \to P_2|\mathbf{K}_2$ that gives the functorial isomorphism $F \cong P_2$ by Proposition 2.6.7. $\square$

Theorem 2.6.14. *The cardinality of a skeleton of the category* **NF** *(***WNF**, **ANF***) is* 2^ω.

Proof. denote by $\mathcal{P}$ a countable set of compact polyhedra containing a unique copy from every topological type of compact polyhedra. The set $\mathcal{P}$ forms a skeleton of the subcategory of compact polyhedra. For each $K, L \in \mathcal{P}$ let $D(K, L)$ be a countable dense subset of $C(K, L)$. We assume that $\mathcal{P}$ and $\bigcup\{D(K, L) \mid K, L \in \mathcal{P}\}$ form a subcategory $\mathcal{C}$ of **Comp**. Using the continuity of normal functors it is easy to check that every normal functor is completely determined by its restriction onto $\mathcal{C}$. It follows from the weight-preserving property that for each $K \in \mathcal{P}$ the space FK belongs to the set of cardinality 2^ω of topological types of metrizable compact Hausdorff spaces, and for each $K, L \in \mathcal{P}$ and $f \in D(K, L)$ the map Ff belongs to the set $C(FK, FL)$ of cardinality 2^ω. Therefore, the cardinality of a family of mutually non-isomorphic (weakly, almost) normal functors is $\leq (2^\omega)^\omega = 2^\omega$.

To complete the proof we construct the family of mutually non-isomorphic normal functors of cardinality 2^ω.

Fix a compact metrizable space K and for each compact Hausdorff space X put

$$F_K X = X^2 \times K / \mathcal{R}_K,$$

where $\mathcal{R}_K$ is the following equivalence relation: $(x, x, k)\mathcal{R}_K(x, x, l)$, $x \in X$, $k, l \in K$. Define F_K as the quotient-functor of the functor $(-)^2 \times K$. It is easily to verify that all the functors F are normal, and F_K and F_L are isomorphic iff K and L are homeomorphic. $\square$

Definition 2.6.15. A normal functor F is called *simple* if for every natural transformation φ of F to a normal functor all maps φX are embeddings.

Theorem 2.6.16. *The only simple functors in* **NF** *are* $\exp$ *and* $\exp_n$, $n \in \mathbb{N}$.

Proof. Show that the functor exp is simple (the case of $\exp_n$ is left to the reader). Let $\varphi\colon \exp \to F$ be a natural transformation. Clearly, we have $\varphi X(\{x\}) \neq \varphi X(\{y\})$ for distinct $x, y \in X$. Suppose that $|X| = 2$ and let $h\colon X \to X$ be a non-identical involution. Then

$$Fh \circ \varphi X(X) = \varphi X \circ \exp h(X) = \varphi(X),$$

and therefore, φX is an embedding.

Consider the case of $\dim X = 0$. It is sufficient to prove the inequality $\varphi X(A) \neq \varphi X(B)$ for every $A, B \subset X$ such that $|A| \geq 2$ and $A \setminus B \neq \varnothing$. Let $x \in A \setminus B$. Taking a map $f\colon X \to 2$ with $f(A) = \{0,1\}$ and $f(B) = \{1\}$, we obtain

$$Ff \circ \varphi X(A) = \varphi 2 \circ \exp f(A) = \varphi 2(\{0,1\}) \neq \varphi 2(\{1\}) = Ff \circ \varphi X(B)$$

that yields the desirable inequality.

Now let $X = I$, $A, B \in \exp I$, $A \setminus B = \varnothing$. If $|A \setminus B| \leq \omega$, consider a map $f\colon I \to I$ such that $B = f^{-1}(0)$. Then $f(A)$ is a countable compact Hausdorff space and $f(B) = \{0\}$. Hence, $\varphi I(A) \neq \varphi I(B)$ by the previous arguments. In the case of $|A \setminus B| > \omega$ we can choose a perfect compact Hausdorff space K in $A \setminus B$ and a map $f\colon I \to I$ with $f(B) = \{0\}$ and $f(K) = f(A) = I$. Thus, $\varphi I(I) \neq \varphi I(\{0\})$, because I is a fixed point of a non-trivial involution $h\colon I \to I$.

Generally, let X be an arbitrary compact Hausdorff space and A, B distinct points of $\exp X$. Then there exists a map $f\colon X \to I$ such that $f(A) \neq f(B)$, and therefore, $\varphi X(A) \neq \varphi X(B)$.

Now prove the second statement of the theorem. Let F be a normal functor which is isomorphic to neither exp nor $\exp_n$. Consider only the case of $\deg F = \infty$. By Theorem 2.5.4 there exists a minimal number $n \in \mathbb{N}$ with $|Fn| > |\exp n|$. Choose distinct points $a, b \in Fn$ such that $\operatorname{supp}_F(a) \neq \operatorname{supp}_F(b) = n$. We construct a quotient-functor F' of F as follows: $F'X = FX/R_X$, where the non-trivial components of the equivalence relation R_X are of the form $\{Fi(a), Fi(b)\}$, $i\colon n \to X$ is an embedding. It is easy to verify normality of F'. Now we have only to note that the natural transformation $F \to F'$ fails the simplicity of F. $\square$

Remark 2.6.17. Normality is an essential property in the previous theorem. Indeed, the functor λ_3 is simple. We leave this as an exercise.

Let F be an almost normal functor and $a \in FX$. For every $Y \in |\mathbf{Comp}|$ let

$$(F/a)Y = \{b \in F(X \times Y) \mid F\mathrm{pr}_1(b) = a\}.$$

We consider F/a as a subfunctor of $F(X \times (-))$. Let $f\colon X \to X'$ be a map and $Ff(a) = a'$. Define the natural transformation $f_*\colon F/a \to F/a'$ by the condition $f_*Y(b) = F(f \times 1_Y)(b)$.

Moreover, for every nonempty closed subset $A \subset FX$ let

$$(F/A)Y = \{b \in F(X \times Y) \mid F\mathrm{pr}_1(b) \in A\}.$$

The functor F/A is a subfunctor of $F(X \times (-))$.

Exercise

1. Let F be a (weakly, almost) normal functor in **Comp**. For every $X \in |\mathbf{Comp}|$ let

   $$F^c X = \{a \in FX \mid \operatorname{supp} a \text{ is contained in a connected component of } X\}.$$

   Show that F^c is a subfunctor of F and F^c is almost normal if so is F.

Problems

1. Is there a universal object in the category **NF** (**WNF**, **ANF**)?
2. A (weakly, almost) normal functor F is called *zero-dimensional* if F preserves the class of zero-dimensional compact Hausdorff spaces. Is every (weakly, almost) normal functor a quotient functor of a (weakly, almost) normal zero-dimensional functor? Is it true for the functors of finite degree?

2.7. Extension of normal functors onto the category of Tychonov spaces

Let $\beta\colon \mathbf{Tych} \to \mathbf{Comp}$ be the Stone-Čech compactification functor. Each Tychonov space X is identified with the naturally embedded subspace in βX.

Recall that a map $f\colon X \to Y$ of Tychonov spaces is *k-covering* if for every compact subspace $B \subset Y$ there is a compact subspace $A \subset X$ with $f(A) = B$.

Definition 2.7.1. A functor $L\colon \mathbf{Tych} \to \mathbf{Tych}$ is called *normal* if L is continuous, preserves weight, embeddings, intersections, preimages, singletons, empty set, and sends k-covering maps to surjections.

Let L be a normal functor in **Tych**, $X \in |\mathbf{Tych}|$ and $a \in LX$. The sets

$$\ker(a) = \bigcap\{A \subset X \mid a \in FA \subset FX\}$$

and

$$\mathrm{supp}(a) = \bigcap\{A \text{ is closed in } X \mid a \in FA \subset FX\}$$

are called the *kernel* and the *support of* a, respectively. It is obvious that $\mathrm{Cl}(\ker(a)) = \mathrm{supp}(a)$.

For every normal functor $F\colon \mathbf{Comp} \to \mathbf{Comp}$ construct the functor $F_\beta\colon \mathbf{Tych} \to \mathbf{Tych}$ by the following manner. For every $X \in |\mathbf{Tych}|$ let

$$F_\beta X = \{a \in F\beta X \mid \mathrm{supp}_F(a) \subset X\} \subset F\beta X.$$

Given a map $f\colon X \to Y$ of Tychonov spaces, we have $F\beta f(F_\beta X) \subset F_\beta Y$. Let $F_\beta f = F\beta f|F_\beta X\colon F_\beta X \to F_\beta Y$. Obviously, F_β is an endofunctor in **Tych**. By definition, F_β is an extension of F onto **Tych**.

Proposition 2.7.2. *Let X be a Tychonov space and bX be a compactification of X. Then the subspace $\{a \in FbX \mid \mathrm{supp}(a) \subset X\}$ of FbX is homeomorphic to $F_\beta X$.*

Proof. Let $g\colon \beta X \to bX$ be a unique extension of the map 1_X. Since F preserves preimages, the map $Fg|F_\beta X$ bijectively maps the space $F_\beta X$ onto its image and

$$(Fg)^{-1}(\{a \in FbX \mid \mathrm{supp}(a) \subset X\}) = F_\beta X$$

(note that $g^{-1}(X) = X$). Besides, the map $Fg|F_\beta X \to Fg(F_\beta X)$, being a restriction of a closed map onto a full preimage, is closed, and, therefore, is a homeomorphism. □

Theorem 2.7.3. *For every normal functor $F\colon \mathbf{Comp} \to \mathbf{Comp}$ the functor $F_\beta\colon \mathbf{Tych} \to \mathbf{Tych}$ is normal.*

Proof. Show that F_β preserves weight. Given $X \in |\mathbf{Tych}|$, embed X into $Y \in |\mathbf{Comp}|$ with $w(X) = w(Y)$. Then

$$w(F_\beta X) \le w(FY) = w(Y) = w(X).$$

Prove that F_β is continuous. Let $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ be an inverse system in **Tych**, $X = \varprojlim \mathcal{S}$. Denote by bX the limit of the inverse system $\beta\mathcal{S} = \{\beta X_\alpha, \beta p_{\alpha\beta}; \mathcal{A}\}$.

Obviously, bX is a compactification of X. Since F is a continuous functor, we have

$$\varprojlim F(\beta\mathcal{S}) = F(\varprojlim \beta\mathcal{S}) = FbX.$$

Show that

$$\varprojlim F_\beta(\mathcal{S}) = \{a \in FbX \mid \operatorname{supp}(a) \subset X\}.$$

If $a' \in FbX$ is such that $a' \in \varprojlim F_\beta(\mathcal{S})$, then, denoting by $\pi_\alpha\colon bX \to \beta X_\alpha$, $\alpha \in \mathcal{A}$, the limit projections of $\beta\mathcal{S}$, we obtain that $\operatorname{supp} F\pi_\alpha(a') \subset X_\alpha$.

Since F preserves supports, we have

$$p_{\alpha\beta}(\operatorname{supp} F\pi_\alpha(a')) = \operatorname{supp} F\pi_\beta(a'),$$

whenever $\alpha \geq \beta$. Now continuity of the functor exp implies the inclusion $\operatorname{supp}(a') \subset X$. By Proposition 2.7.2, we obtain that F_β is a continuous functor. □

The following proposition shows that the described extension F_β is a minimal normal extension of a normal functor F: **Comp** → **Comp**.

Proposition 2.7.4. *Let $\widetilde{F}$:* **Tych** → **Tych** *be a normal functor which is an extension of a normal functor F:* **Comp** → **Comp**. *Then F_β is a subfunctor of $\widetilde{F}$.*

Proof. Let $X \in |$**Tych**$|$ and $a \in F_\beta X$. Then there exists a compact subset $A \subset X$ such that $a \in F_\beta A = FA = \widetilde{F}A \subset \widetilde{F}X$, whence $F_\beta X \subset \widetilde{F}X$, the latter embedding being natural by X. □

Proposition 2.7.5. *Let $\varphi\colon F \to F'$ be a natural transformation of normal functors F, F':* **Comp** → **Comp**. *Then there exists a unique natural transformation $\varphi_\beta\colon F_\beta \to F'_\beta$ such that $\varphi_\beta X = \varphi X$ for every $X \in |$***Comp***$|$.*

Proof. Uniqueness. Let X be a Tychonov space, $a \in F_\beta X$, $\operatorname{supp}(a) = A$. Denoting by $i\colon A \to X$ the identical embedding, we obtain for every natural transformation $\bar\varphi\colon F_\beta \to F'_\beta$

$$\bar\varphi X \circ F_\beta i(a) = F'_\beta i \circ \bar\varphi A(a) = F'_\beta i \circ \varphi A(a),$$

hence, $\bar\varphi$ is completely determined by φ.

Let $j\colon A \to \beta X$ be an embedding. Then

$$F'j \circ \varphi A(a) = \varphi\beta X \circ Fj(a),$$

therefore, $\varphi\beta X(F_\beta X) \subset F'_\beta X$. Put

$$\varphi_\beta X = \varphi\beta X | F_\beta X\colon F_\beta X \to F'_\beta X.$$

Clearly, $\varphi_\beta = (\varphi_\beta X)$ is a natural transformation that extends φ. □

Theorem 2.7.6. *If a normal functor $F\colon$ **Tych** $\to$ **Tych** is a functor with compact supports and $F($**Comp**$) \subset$ **Comp**, then we have $F = (F|$**Comp**$)_\beta$.*

Proof. For every Tychonov space X and $a \in FX$ we have that $a \in FA$ for some compact Hausdorff space $A \subset X$. Hence, $a \in (F|\mathbf{Comp})_\beta X$. Therefore, $F \subset (F|\mathbf{Comp})_\beta$ and by Proposition 2.7.4, $F = (F|\mathbf{Comp})_\beta$. □

Similarly to the compact case, the normal functors in **Tych** and their natural transformation form a category $\mathbf{NF}_{\mathbf{Tych}}$. Obviously, $(-)_\beta$ is a functor which embeds **NF** into $\mathbf{NF}_{\mathbf{Tych}}$.

For normal functors $F, F'\colon$ **Comp** $\to$ **Comp** we have

$$\begin{aligned}(F'F)_\beta X =&\{a \in F'F_\beta X \mid \mathrm{supp}_{F'F}(a) \subset X\}\\ \subset&\Big\{a \in F'F_\beta X \mid \mathrm{Cl}\Big(\bigcup\{\mathrm{supp}_F(b) \mid b \in \mathrm{supp}_{F'}(a)\}\Big) \subset F_\beta X\Big\}\\ =&F'_\beta F_\beta X.\end{aligned}$$

The embedding $(F'F)_\beta X \hookrightarrow F'_\beta F\beta X$ is natural with respect to X. We write $(F'F)_\beta = F'_\beta F_\beta$, whenever this embedding is a functorial isomorphism.

Theorem 2.7.7. *If F is a normal functor with continuous supports, then $(F'F)_\beta = F'_\beta F_\beta$ for every normal functor F.*

Proof. For every $a \in F'_\beta F_\beta X \subset F'F(\beta X)$ we have

$$\begin{aligned}\mathrm{supp}_{F'F}(a) = \mathrm{Cl}\Big(\bigcup\{\mathrm{supp}_F(b) \mid b \in \mathrm{supp}_{F'}(a)\}\Big) =\\ = \bigcup\{\mathrm{supp}_F(b) \mid b \in \mathrm{supp}_{F'}(a)\} \subset X,\end{aligned}$$

and therefore, $a \in (F'F)_\beta X$. □

Definition 2.7.8. A functor $L\colon \mathbf{Tych} \to \mathbf{Tych}$ is called *countably co-continuous* if it commutes with colimits of countable direct systems of compact Hausdorff spaces and embeddings.

Proposition 2.7.9. *The functor* $\exp = \exp_\beta$ *is countably co-continuous.*

We will need the following

Lemma 2.7.10. *Let* $A \in \langle B_1, \dots, B_n \rangle \subset U$, *where* $A, B_1, \dots, B_n \in \exp X$ *and* U *open in a compact Hausdorff space* X. *Then there exist open sets* $V_1, \dots, V_n$ *in* X *such that* $B_i \subset V_i$, $1 \le i \le n$, *and* $\langle V_1, \dots, V_n \rangle \subset U$.

The proof follows from the fact that each B_i is contained in the closure (in $\exp X$) of the set $\{C \in \exp X \mid B \subset \operatorname{Int} C\}$ and from the continuity of the map $\alpha\colon (\exp X)^n \to \exp^2 X$, $\alpha(A_1, \dots, A_n) = \langle A_1, \dots, A_n \rangle$ that can be checked immediately.

Proof of Proposition 2.7.9. Let $X = \varinjlim\{X_i\}$ (we assume that $X_i \subset X_j \subset X$ for $i < j$). By h we denote the natural continuous map from $\varinjlim\{\exp X_i\}$ to $\exp X$. It is obvious that h is bijective thus, it is sufficient to show the continuity of h^{-1}.

Let $A \in \varinjlim\{\exp X_i\}$, then $A \in \exp X_k \subset \varinjlim\{\exp X_i\}$ for some k. Let U be a neighborhood of A in the space $\varinjlim\{\exp_i\}$ and let $U_i = U \cap \exp X_i$. Using Lemma 2.7.10 we construct by induction on $i \ge k$ the family $\{V_{i1}, \dots, V_{is}\}$ of open sets in X satisfying: $V_{i'j} \subset V_{ij}$ if $i' \le i$, $1 \le j \le s$, and

$$A \in \langle V_{i1}, \dots, V_{is} \rangle \subset U_i \subset \exp X_i.$$

Put $V_j = \bigcup\{V_{ij} \mid i \in \mathbb{N}\}$, then

$$h(A) = A \in \langle V_1, \dots, V_s \rangle \subset \exp\bigl(\varinjlim\{X_i\}\bigr)$$

and $h^{-1}(\langle V_1, \dots, V_s \rangle) \subset U$. Openness of $V_1, \dots, V_s$ in $\varinjlim\{X_i\}$ implies continuity of h^{-1}. □

Theorem 2.7.11. *The following conditions are equivalent for every normal functor* $F\colon \mathbf{Comp} \to \mathbf{Comp}$:

1) *F is a functor with continuous supports;*
2) *F_β is countably co-continuous;*
3) *F_β preserves the class of locally compact spaces.*

Proof. 1)⇒2). Since the map supp: $F(\beta X) \to \exp(\beta X)$ is closed, we see that the same is its restriction supp: $F_\beta X \to \exp X$. Continuity of supports implies also that the map supp is perfect. Now let $X = \varinjlim\{X_i\}$, where X_i are compact Hausdorff spaces. By Proposition 2.7.9, $\exp X = \varinjlim\{\exp X_i\}$ and, therefore, $\exp X$ is a k-space. Hence FX is also a k-space as a preimage of a k-space trough perfect map. Note that for each compact Hausdorff space $A \subset F_\beta X$ there exists j such that $\mathrm{supp}(a) \subset \exp X$ for each $a \in A$, and the preimage-preserving property implies that $A \subset F_\beta X$. But then $F_\beta X = \varinjlim\{F_\beta X_j\}$.

2)⇒3). Suppose F is not a functor with continuous supports. Put $X_i = Q \times [0, i]$, then $\varinjlim\{X_i\} = Q \times [0, \infty)$. It is easy to deduce the existence of a sequence $\{a_i \mid i \in \mathbb{N}\} \subset F_\beta X$ such that $\mathrm{supp}(a_i) \cap X_i \neq \varnothing$ for each i and

$$\lim_{i\to\infty}\{a_i\} = a \in F_\beta X.$$

Assuming countable co-continuity of F_β, we obtain $F_\beta X = \varinjlim\{F_\beta X_i\}$. But $A = \{a_i \mid i \in \mathbb{N}\} \cup \{a\}$ is a compact Hausdorff space in $F_\beta X$ with $A \cap F_\beta X_j \neq \varnothing$ for each $j \in \mathbb{N}$, so we get a contradiction.

1)⇒3). Let X be a locally compact space. It is easy to see that $\exp X$ is locally compact as well. But then $F_\beta X$ is also locally compact as the preimage of a locally compact with respect to perfect map supp: $F_\beta X \to \exp X$.

3)⇒1). Let F_β preserves the class of locally compact spaces. Then the space $F_\beta\mathbb{R}$ is locally compact and hence is open in $F(\beta\mathbb{R})$. Assuming that the functor F fails to be a functor with continuous supports it is easy to construct the sequence $\{a_i \mid i \in \mathbb{N}\} \subset F(\beta\mathbb{R})$ such that

$$\mathrm{supp}(a_i) \cap (\beta\mathbb{R} \setminus \mathbb{R}) \neq \varnothing \text{ and } \lim_{i\to\infty}\{a_i\} = a \in F_\beta\mathbb{R}.$$

This contradicts to the openness of $F_\beta\mathbb{R}$ in $F(\beta\mathbb{R})$. □

Definition 2.7.12. A pair (F', i) is called a *compactification* of a functor F: **Comp** → **Tych** if F': **Comp** → **Comp** is a functor and $i\colon F \to F'$ is a natural transformation such that, for every $X \in |\mathbf{Comp}|$ the map iX is an embedding and the space FX is dense in $F'X$.

Denote by $\mathbf{Comp}^\infty$ the full subcategory of **Tych** whose objects are k_ω-spaces, i. e. the countable direct limits of sequences of compact Hausdorff spaces and inclusion maps. Given a (weakly, almost) normal

functor F in **Comp** we denote by F^∞ its extension onto **Comp**$^\infty$ defined by the rule

$$F^\infty(\varinjlim X_i) = \varinjlim FX_i, \quad F^\infty f = \varinjlim F(f|X_i)$$

(here $f\colon X = \varinjlim X_i \to Y = \varinjlim Y_j$ is a map and $f|X_i$ maps X_i onto $f(X_i)$).

More generally, if $F^{(1)} \subset F^{(2)} \subset \dots$ is a direct sequence of (weakly, almost) normal functors in **Comp**, we can define its direct limit

$$F^{(\infty)}\colon \mathbf{Comp}^\infty \to \mathbf{Comp}^\infty$$

by $F^{(\infty)}\colon \varinjlim X_i = \varinjlim F^{(i)}X_i$.

A special case of this situation is obtained if there is a (weakly, almost) normal functor F in **Comp** and $F^{(i)} = F_i$. We use the notation F_∞ for $\varinjlim F_i$.

Exercises

1. For normal functors in **Tych** prove counterparts of properties of normal functors from Section 2.6.
2. Similarly to the definition of normal functor in **Tych** one can define the notion of (weakly, almost) normal functor in **Tych**. Prove that there is no weakly normal extension of the functor Pr^3 onto **Tych**.
3. Let $F\colon \mathbf{Comp} \to \mathbf{Comp}$ be a functor for which there exists a natural embedding $\eta\colon \mathrm{Id} \to F$. Prove that there is at most countable family of mutually nonisomorphic functors of the form $\varinjlim\{F^i, \varphi_i\}\colon \mathbf{Comp} \to \mathbf{Comp}^\infty$, where $\varphi_i \in \{F^i\eta, \eta F^i\}$. Show that if SP^2 is a subfunctor of F then at least two of these functors are nonisomorphic.

Problems

1. Find an uncountable family of extensions of the functor $(-)^\omega\colon \mathbf{Comp} \to \mathbf{Comp}$ onto the category **Tych**. Is there such a family of cardinality continuum?
2. Is there a maximal normal extension onto the category **Tych** of a normal functor in the category **Comp**?
3. Find an intrinsic characterization of the functor $(-)^\omega$ in **Tych**.

2.8. Functorial operators extending functions and (pseudo)metrics

The aim of this section is to investigate connections between functors and spaces of continuous functions and (pseudo)metrics.

Let F be an endofunctor in **Comp** for which there exists a natural embedding $\eta = \eta_F\colon \mathrm{Id} \to F$.

Definition 2.8.1. A collection of maps

$$T = (T_X\colon C(X) \to C(FX))_{X \in |\mathbf{Comp}|}$$

is called a functorial operator extending functions (FOEF) if for every morphism $f\colon X \to Y$ in **Comp** the diagram

$$\begin{tikzcd} C(X) \arrow[d, "T_X"'] & C(Y) \arrow[l, "(-)\circ f"'] \arrow[d, "T_Y"] \\ C(FX) & C(FY) \arrow[l, "(-)\circ Ff"'] \end{tikzcd}$$

is commutative.

We will specify this notion by requiring that the maps T_X are continuous, linear, regular, multiplicative, order-preserving etc. For $t \in \mathbb{R}$ we denote by c_t the constant function with the range $\{t\}$. Clearly, $T_X(c_t) = c_t$ for every FOEF $T = (T_X)$.

If $T = (T_X)$ is an FOEF for a functor F and a natural transformation $\alpha F' \to F$ satisfies the condition $\alpha\eta_{F'} = \eta_F$, then $T' = (T'_X)$, where $T'X(\varphi) = \alpha X \circ T_X(\varphi)$, is an FOEF for T'.

Examples. 1) For the symmetric power functor SP^{2n+1} by the formula

$$T_X(\varphi)[x_0, \dots, x_{2n}] = (\varphi(x_0)\dots\varphi(x_{2n}))^{1/(2n+1)}.$$

Obviously, $T = (T_X)$ is a continuous multiplicative (i. e. preserving the pointwise multiplication) FOEF.

2) For the hyperspace functor exp define the FOEFs $T_{\max}$ and $T_{\min}$ by the formulae

$$T_{\max X}(\varphi)(A) = \max\{\varphi(x) \mid x \in A\},\ T_{\min X}(\varphi)(A) = \min\{\varphi(x) \mid x \in A\}.$$

The FOEFs $T_{\max}$ and $T_{\min}$ are regular (i.e. they do not increase sup-norm of functions) and order-preserving.

3) For the inclusion hyperspace functor G define the FOEF $T_{\max\min}$ by the formula

$$T_{\max\min X}(\mathcal{A}) = \max\{\min\{\varphi(x) \mid x \in A\} \mid A \in \mathcal{A}\}$$

and the FOEF $T_{\min\max}$ by interchanging min and max in the above formula.

Proposition 2.8.2. *There is no multiplicative FOEF for the functor SP^{2n}.*

Proof. Suppose the contrary and let $T = (T_X)$ be such an operator. Consider $X = \{x_0, x_1, \dots, x_{2n-1}\}$, $|X| = 2n$. Denote by $\varphi \in C(SP^{2n}X$ the function defined by $\varphi(x_{2i}) = 0$, $\varphi(x_{2i+1}) = 1$ for $i = 0, 1, \dots, n-1$. Denote by $h\colon X \to X$ the map that permutes x_{2i} and x_{2i+1}, for every $i = 0, 1, \dots, n-1$. Let

$$T_X(\varphi)[x_0, x_1, \dots, x_{2n-1}] = a \in \mathbb{R}.$$

Since $\varphi \cdot \varphi = c_1$ (by $\cdot$ the pointwise multiplication is denoted), we have $a^2 = 1$.

On the other hand,

$$\begin{aligned} -a =& T_X(c_{-1} \cdot \varphi)[x_0, x_1, \dots, x_{2n-1}] = T_X(\varphi \circ h)[x_0, x_1, \dots, x_{2n-1}] \\ &= T_X(\varphi)[x_0, x_1, \dots, x_{2n-1}] = a, \end{aligned}$$

which gives a contradiction. □

Corollary 2.8.3. *There is no multiplicative FOEF for the functors* exp, $\exp_n$, $n \geq 2$, P, P_n, $n \geq 2$, G, N_k, $k \geq 2$.

Proof. It is sufficient to remark that each of these functors contains a subfunctor isomorphic to SP^2. □

Proposition 2.8.4. *The FOEFs $T_{\max\min}$ and $T_{\min\max}$ coincide if they are considered as FOEFs for the subfunctor λ of G. The functor λ is a maximal (with respect to inclusion) subfunctor of G with the property that the FOEFs $T_{\max\min}$ and $T_{\min\max}$ coincide if they are considered as FOEFs for F*

Proof. Let $\mathcal{M} \in \lambda X$. Since $\mathcal{M}$ is linked, we have

$$\begin{aligned} \alpha &= \max\{\min\{\varphi(x) \mid x \in M\} \mid M \in \mathcal{M}\} \\ &\leq \min\{\max\{\varphi(x) \mid x \in M\} \mid M \in \mathcal{M}\} = \beta \end{aligned}$$

for every $\varphi \in C(X)$. Suppose that $\alpha < \beta$ for some $\mathcal{M} \in \lambda X$ and $\varphi \in C(X)$. Consider the closed in X subsets

$$A_0 = \{x \mid \varphi(x) \leq \frac{\alpha + \beta}{2}\}, \ A_1 = \{x \mid \varphi(x) \geq \frac{\alpha + \beta}{2}\}.$$

Since $\mathcal{M}$ is a maximal linked system, one of the sets A_0, A_1, say A_0, is contained in $\mathcal{M}$. Then $\max\{\varphi(x) \mid x \in A_0\} \leq (\alpha + \beta)/2 < \beta$ and hence $\min\{\max\{\varphi(x) \mid x \in M\} \mid M \in \mathcal{M}\} < \beta$, a contradiction.

Now, suppose $\mathcal{A} \in GX \setminus \lambda X$. If $\mathcal{A} \notin NX$, choose $A_0, A_1 \in \mathcal{A}$ such that $A_0 \cap A_1 = \varnothing$. If $\mathcal{A} \in NX \setminus \lambda X$, choose $A_0, A_1 \in \exp X$ such that $A_0 \cap A_1 = \varnothing$ and $\{A_i\} \cup \mathcal{A} \in NX$, $i = 0, 1$. In both cases, choose $\varphi \in C(X, [0,1]$ such that $\varphi|A_i = i$, $i = 0, 1$. Then $\min\{\max\{\varphi(x) \mid x \in A\} \mid A \in \mathcal{A}\} = 0$ and $\max\{\min\{\varphi(x) \mid x \in A\} \mid A \in \mathcal{A}\} = 1$.

□

An operator $u\colon C(X) \to C(Y)$ is called *orthogonal* if the condition $\min\{\varphi, \psi\} = c_0$ (respectively $\max\{\varphi, \psi\} = c_0$) implies the condition $\min\{u(\varphi), u(\psi)\} = c_0$ (respectively $\max\{u(\varphi), u(\psi)\} = c_0$).

Proposition 2.8.5. *The FOEP $T = T_{\min\max}$ for λ is orthogonal.*

Proof. Let $\varphi, \psi \in C(X)$ and $\min\{\varphi, \psi\} = c_0$. Suppose that for some $\mathcal{M} \in \lambda X$ we have $T_X(\varphi)(\mathcal{M}) > 0$. Then $\varphi^{-1}(0) \notin \mathcal{M}$. Since $\varphi^{-1}(0) \cup \psi^{-1}(0) = X$, by maximality of $\mathcal{M}$ we see that $\psi^{-1}(0) \in \mathcal{M}$ and consequently $T_X(\psi)(\mathcal{M}) = 0$, i. e. $\min\{T_X(\varphi), T_X(\psi)\} = c_0$. □

For any $X \in |\mathbf{Comp}|$ denote by $\mathcal{P}(X)$ the set of all continuous pseudometrics on X. Given a map $f\colon X \to Y$ of metrizable compact Hausdorff spaces, we denote by $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ the map defined by $f^*(d) = f \circ (d \times d)$.

Let $F\colon \mathbf{Comp} \to \mathbf{Comp}$ be a functor for which there exists a natural transformation $\eta\colon \mathrm{Id} \to F$. A collection

$$T = (T_X\colon \mathcal{P}(X) \to \mathcal{P}(FX))_{X \in |\mathbf{Comp}|}$$

is called a *functorial operator extending pseudometrics* (FOEP) if the following holds:

1) $\eta X^* \circ T_X(d) = d$, for every $d \in \mathcal{P}(X)$;

2) for every morphism $f\colon X \to Y$ the diagram

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "T_Y"] \arrow[d, "f^*"'] & \mathcal{P}(FY) \arrow[d, "(Ff)^*"] \\ \mathcal{P}(X) \arrow[r, "T_X"] & \mathcal{P}(FX) \end{tikzcd}$$

is commutative. An FOEP $T = (T_X)_{X \in |\mathbf{Comp}|}$ is called *linear* if all T_X preserve linear operations in the cones of pseudometrics.

If $T = (T_X)$ is an FOEP for a functor F and a natural transformation $\alpha F' \to F$ satisfies the condition $\alpha\eta_{F'} = \eta_F$, then $T' = (T'_X)$, where $T'X(d) = (\alpha X \times \alpha X) \circ T_X(d)$, is an FOEP for T'.

Proposition 2.8.6. *Let $n \in \mathbb{N}$. There exists a linear regular FOEP for the Hartman-Mycielski functor HM_n.*

Proof. Let (X, d) be a compact metric space. Define the metric $T_X(d)$ on HM_nX by the formula

$$T_X(d)(\alpha, \beta) = \int_0^1 d(\alpha(t), \beta(t))dt, \quad \alpha, \beta \in HM_nX.$$

It can be easily checked that $T = (T_X)$ is a required FOEP. □

Proposition 2.8.7. *There exists no linear FOEP for the symmetric power functor SP^2.*

Proof. We need some auxiliary results.

Lemma 2.8.8. *Let $X = \{x_1, \dots, x_m\}$ be a finite set. Suppose that a_{ij}, $1 \le i \le j \le m$, are the reals such that for every metric d on X the equality*

$$\sum_{i<j} a_{ij} d(x_i, x_j) = 0. \tag{2.7}$$

Then $a_{ij} = 0$ for every i, j.

Proof. Let d_1 be the metric on X with all nonzero distances equal to 1, and d_2 the metric coinciding with d_1 with the exception $d(x_i, x_j) = 2$. Subtracting the corresponding equalities (2.7), we obtain $a_{ij} = 0$. □

Lemma 2.8.9. *Let $X = \{x_1, \dots, x_m\}$ be a finite set, $m > 2$. For every i, j let E_{ij} denote the pseudometric on X such that $E_{ij}(x_i, x_j) = 0$ and all other distances between distinct points are 1. Then for any $d \in \mathcal{P}(X)$ there exist $e_{ij} \in \mathbb{R}$ such that $d = \sum_{i<j} E_{ij}$.*

Now let us return to the proof of the theorem. Suppose that $X = \{x_1, x_2, y_1, y_2\}$, where all the points are different. If T is a linear FOEP for SP^2 and d a metric on X, we have by Lemma 2.8.9

$$T_X(d)([x_1, x_2], [y_1, y_2]) = \sum_{i,j=1}^{2} a_{ij} d(x_i, y_j) + bd(x_1, x_2) + cd(y_1, y_2), \tag{2.8}$$

for some $a_{ij}, b, c \in \mathbb{R}$.

Swapping x's and y's in (2.8) and comparing, we obtain

$$d(x_1, x_2)(b-c) + d(y_1, y_2)(c-b) + \sum_{i,j=1}^{2} d(x_i, y_j)(a_{ij} - a_{ji}) = 0.$$

Then, by Lemma 2.8.8, $b = c$ and $a_{ij} = a_{ji}$.

Since $T_X(d)([x_1, x_2], [x_1, x_2]) = 0$, we obtain $a_{ij} = a_{ji} = -b = -c$. Besides, because of symmetricity of (2.8) with respect to y_1 and y_2, we see that $a_{11} = a_{12} = a_{22}$.

Finally,

$$\begin{aligned} T_X(d)([x,x],[y,y]) &= 4a_{11}d(x,y), \\ T_X(d)([x,x],[x,y]) &= a_{11}d(x,y), \\ T_X(d)([x,y],[y,y]) &= a_{11}d(x,y). \end{aligned}$$

To satisfy the triangle inequality we must have $a_{11} = 0$. Then $d(x,y) = T_X(d)([x,x],[y,y]) = 2a_{11}d(x,y) = 0$ which is not the case for every d. □

Corollary 2.8.10. *There is no linear FOEP for the functors* $\exp_n$, P_n, $n \geq 2$, $\exp$, P.

Proof. This follows for the fact that there exist natural embeddings if SP^2 into these functors and from Proposition 2.8.7. □

Let **Metr** denote the category of compact metric spaces and nonexpanding maps. By $U\colon$ **Metr** $\to$ **Comp** we denote the forgetful functor. Given a functor $F\colon$ **Comp** $\to$ **Comp** we say that a functor $\bar{F}\colon$ **Metr** $\to$ **Metr** is a *lifting* of F if $U\bar{F} = FU$. If there is a natural transformation $\eta\colon \mathrm{Id} \to F$, a lifting $\bar{F}$ is called *regular* whenever $\eta U(X,d)$ is an isometric embedding of (X,d) into $\bar{F}(X,d)$, for every $(X,d) \in |$**Metr**$|$. One of the approaches to the problem of lifting of a functor F onto the category **Metr** consists in finding FOEPs $T = (T_X)$ that preserve metrics (i. e. $T_X(d)$ is a metric if so is d).

Exercises

1. Suppose F is a normal functor F for which there exists a linear FOEP. Is there a natural transformation $F \to \mathrm{Id}$?
2. Construct FOEPs for the functors of "words of length $\leq n$" in the free topological (Abelian) groups.

2.9. Multiplicativity

Definition 2.9.1. A functor is called *(finitely) multiplicative* if it preserves (finite) products. More precisely, given a product $X = \prod\{X_\alpha \mid \alpha \in \Gamma\}$, for a functor F we have the canonical map

$$h = (F\,\mathrm{pr}_\alpha)_{\alpha\in\Gamma} \colon FX \to \prod\{FX_\alpha \mid \alpha \in \Gamma\}$$

(here $\mathrm{pr}_\alpha \colon X \to X_\alpha$ are the projections). The functor F is multiplicative if h is a homeomorphism.

Recall that for a normal functor F and a point $a \in FX$ by F_a denote the following subfunctor of F:

$$F_aY = \bigcap\{F'Y \mid F' \text{ is a normal subfunctor of } F \text{ and } a \in F'X\}.$$

Lemma 2.9.2. *If $F\colon$ **Comp** $\to$ **Comp** is a normal functor for which the map $(F\,\mathrm{pr}_1, F\,\mathrm{pr}_2)\colon F(Q\times Q) \to FQ \times FQ$ is a homeomorphism, then F is multiplicative.*

Proof. For compact metrizable spaces X, Y, supposing that $X \subset Q$, $Y \subset Q$, by the properties of preserving intersections and preimages of F we obtain that the map $(F\,\mathrm{pr}_1, F\,\mathrm{pr}_2)\colon F(X\times Y) \to FX \times FY$ is a homeomorphism. Hence, F preserves finite products of compact metrizible spaces. Finally, by continuity of F we see that it is multiplicative. ☐

Definition 2.9.3. A normal functor F is called *profinitely power* if for every $a \in F_\omega X$ the isomorphism $F_a \cong (-)^{\deg(a)}$ holds.

Definition 2.9.4. A weakly normal functor is *weakly bicommutative* if for every $a \in FX$, $b \in FY$ there exists $c \in F(X \times Y)$ such that $F\,\mathrm{pr}_1(c) = a$, $F\,\mathrm{pr}_2(c) = b$.

Lemma 2.9.5. *Every finitely multiplicative normal functor is profinitely power and weakly bicommutative.*

Proof. It is sufficient to prove that F is profinitely power. Let $a \in F_\omega X$, $\deg(a) = n \in \mathbb{N}$. Without restricting generality, we can suppose that $X = n$. Then

$$F_aY = \{Ff(a) \mid f \in C(n, Y) = Y^n\}.$$

By multiplicativity of F, we see that for every $b \in FY$ there exists a unique point $c \in F(n \times Y)$ with $F\operatorname{pr}_1(c) = a$, $F\operatorname{pr}_2(c) = b$. Hence, every point $b \in FY$ determines a unique map $f \in C(n, Y) = Y^n$ such that $Ff(a) = b$. Being a restriction of the Basmanov map, the map $\xi Y: C(n, Y) = Y^n \to F_a Y$, $\xi Y(f) = Ff(a)$, is continuous. Therefore, it is a homeomorphism. Clearly, $\xi Y = (\xi Y): (-)^n \to F_a$ is a functorial isomorphism. □

Lemma 2.9.6. *Let F be a normal profinitely power weakly bicommutative functor. Then for every zero-dimensional compact Hausdorff spaces X, Y, points $a \in FX$, $b \in FY$, and $c, c' \in F(X \times Y)$ we have* $\operatorname{supp}(c) = \operatorname{supp}(c')$ *whenever* $F\operatorname{pr}_1(c) = F\operatorname{pr}_1(c') = a$, $F\operatorname{pr}_2(c) = F\operatorname{pr}_2(c') = b$.

Proof. For finite discrete spaces X, Y by bicommutativity of F there exists $d \in F(X \times Y)$ with $c \in F_d(X \times Y)$, $c' \in F_d(X \times Y)$. Since $F_d \cong (-)^{\deg(d)}$, we have $c = c'$. In particular, obtain that $\operatorname{supp}(c) = \operatorname{supp}(c')$.

It is known that every zero-dimensional compact Hausdorff space can be represented as the limit of an inverse system of finite compact Hausdorff spaces. Thus, continuity of F implies the general case. □

Theorem 2.9.7. *Every normal profinitely power weakly bicommutative functor is a power functor.*

Proof. Suppose first that $\deg F = n < \infty$. Choose $a \in Fn$ with $\deg(a) = n$. By the weak bicommutativity of F, for every compactum X and point $b \in FX$ there exists a point $c \in F(X \times Y)$ such that $F\operatorname{pr}_1(c) = a$, $F\operatorname{pr}_2(c) = b$. Then the map $\operatorname{pr}_1 | \operatorname{supp}(c): \operatorname{supp}(c) \to n$ is one-to-one. Hence, the set $\operatorname{supp}(c)$ is the graph of a map $f: n \to X$ such that $Ff(a) = b$. Therefore, the family

$$\xi X: X^n = C(n, X) \to FX, \ \xi X(f) = Ff(a), \ X \in |\mathbf{Comp}|,$$

forms a functorial isomorphism $\xi: (-)^n \to F$.

Now consider the case $\deg F = \infty$. Denote by $\leq_X$ the following pre-order relation on $F_\omega X$: for $a_1, a_2 \in F_\omega X$ we have $a_1 \leq_X a_2$, whenever there exists a map $f: \operatorname{supp}(a_2) \to \operatorname{supp}(a_1)$ such that $Ff(a_2) = a_1$. Remark that if $a_1 \leq a_2$ then $F_{a_1} \subset F_{a_2}$.

By the weak bicommutativity of F, for every infinite compact Hausdorff space X and $a_1, a_2 \in F_\omega X$ there exists $b \in F_\omega X$ such that $b \geq a_1$,

$b \geq a_2$. Moreover, we can choose every map $f_i\colon \mathrm{supp}(b) \to \mathrm{supp}(a_i)$ with $Ff_i(b) = a_i$, $i = 1, 2$, to be a retraction.

Let 2^ω be the Cantor set and $\{b_i \mid i < \omega\}$ a countable dense subset of $F_\omega(2^\omega)$ (and $F(2^\omega)$). Choose a subset $\{a_i \mid i < \omega\} \subset F_\omega(2^\omega)$ such that $b_i \leq a_i$, $i < \omega$, and $a_0 \leq a_1 < \dots$. We have for every metrizable compact Hausdorff space X that the set $\bigcup\{F_{a_i}X \mid i < \omega\}$ is dense in FX. Indeed, if $f\colon 2^\omega \to X$ is onto then

$$\bigcup\{F_{a_i} \mid i < \omega\} = Ff(\bigcup\{F_{a_i}(2^\omega) \mid i < \omega\}) \supset Ff(\{b_i \mid i < \omega\}),$$

and the latter subset is dense in FX.

Putting $A_i = \mathrm{supp}(a_i)$, obtain a sequence of finite sets

$$A_0 \subset A_1 \subset A_2 \subset \dots .$$

There exists a sequence of retractions $r_{ij}\colon A_i \to A_j$, $i \geq j$, for which $Fr_{ij}(a_i) = a_j$.

Set $A = \bigcup\{A_i \mid i < \omega\}$. Let $r_i\colon A \to A_i$ be a retraction such that $r_i|A_i = r_{ji}$, $j \geq i$. Denote by $\tilde{r}\colon \beta A \to A_i$ an extension of r_i onto the Stone-Čech compactification βA of discrete space A. Let $a \in F\beta A$ be a limit point of $\{a_i \mid i < \omega\}$. Then $F\tilde{r}_i(a) = a_i$.

Let X be a finite discrete space and $d \in FX$. There exists a sequence $\{d_i \mid i < \omega\}$, $d_i \in F_{a_i}X$ such that $\lim_{i\to\infty} d_i = d$. Choose points $c_i \in F(\beta A \times X)$ such that $F\,\mathrm{pr}_1(c_i) = a_i$, $F\,\mathrm{pr}_2(c_i) = d_i$. Let $c \in F(\beta A \times X)$ be a limit point of $\{c_i \mid i < \omega\}$ with $F\,\mathrm{pr}_1(c) = a$. Then $F\,\mathrm{pr}_2(c) = d$. By Lemma 2.9.6 we have $\mathrm{supp}(c') = \mathrm{supp}(c)$ for every $c' \in F(\beta A \times X)$ such that $F\,\mathrm{pr}_1(c') = a$, $F\,\mathrm{pr}_2(c') = d$.

Note that the set $\mathrm{supp}(c_i) \cap (\{y\} \times X)$, $i < \omega$, $y \in A$, is either empty or a singleton. Hence, $\mathrm{supp}(c_i) \cap (\{y\} \times A)$ is a singleton (because the map $\mathrm{supp}\colon FX \to \exp X$ is lower semicontinuous and $\mathrm{pr}_1(\mathrm{supp}(c)) = \beta A$). Denote by $f_y(d)$ a unique point of X with $(y, f_y(d)) \in \mathrm{supp}(c)$.

The point $f_y(d)$ is completely determined by c. Since the map supp is lower semicontinuous, the map $f_y\colon FX \to X$ is continuous.

Define a map $\xi X : FX \to X^A$ by the following condition:

$$\mathrm{pr}_y(\xi X(d)) = f_y(d), \ d \in FX, \ y \in A.$$

This map is continuous for every zero-dimensional compact metrizable space X.

Show that ξX is an injective map for every finite X. Indeed, otherwise, there exists $c, c' \in F(\beta X \times A)$ such that $\operatorname{supp}(c) = \operatorname{supp}(c')$, the set $\operatorname{supp}(c) \cap (\{y\} \times X)$ is a singleton for every $y \in A$, $F\operatorname{pr}_1(c) = F\operatorname{pr}_1(c') = a$, and $F\operatorname{pr}_2(c) \neq F\operatorname{pr}_2(c')$. Let $f\colon A \to X$ be a map determined by the graph $\operatorname{supp}(c) \cap (A \times X)$, $\tilde{f}\colon \beta A \to X$ its extension. There exists a retraction $\varrho\colon \beta A \to \beta A$ with $\varrho(\beta A) \subset A$ and $\tilde{f} \circ \varrho = \tilde{f}$. Then the supports of $F(\varrho \times 1_X)(c)$ and $F(\varrho \times 1_X)(c')$ are equal. Moreover, they are bijectively projected onto the support of $F\varrho(a)$. Since X is finite, we have

$$F(\varrho \times 1_X)(c) = (\varrho \times 1_X)(c').$$

But

$$F\operatorname{pr}_2 \circ F(\varrho \times 1_X)(c) = F\operatorname{pr}_2(c) \neq F\operatorname{pr}_2(c') = F\operatorname{pr}_2 \circ F(\varrho \times 1_X)(c').$$

Obviously, ξX is onto, thus ξX is a homeomorphism for every finite X. By the naturality on X of the map ξX and the continuity of F and $(-)^A$, we see that ξX is a homeomorphism for every zero-dimensional X.

Define a natural transformation $\tau\colon (-)^A \to F$ by $\tau X(f) = F\tilde{f}(a)$, where $f \in X^A = C(A, X)$ and $\tilde{f}\colon \beta A \to X$ is an extension of f. Clearly, τ is inverse to ξ on the category $\mathbf{Comp}_0$ of zero-dimensional compact Hausdorff spaces.

Show that τ is a functorial isomorphism. By Proposition 2.6.2 it is sufficient to prove for this that the map τQ is a homeomorphism.

Obviously, that τQ is surjective. Suppose that it is not injective. Let $x = (x_0, x_1, \dots)$, $y = (y_0, y_1, \dots)$ be points such that $\tau Q(x) = \tau Q(y)$ and $x_k \neq y_k$ for some $k \in \omega$.

Let $z \in Q \setminus (\{x_0, \dots, x_k\} \cup \{y_0, \dots, y_k\})$. For every $n < \omega$ denote by $f_n\colon Q \to Q$ a map such that:

1) $f_n|(\{x_0, \dots, x_k\} \cup \{y_0, \dots, y_k\})$ is the identity;
2) $f_n(\{x_{k+1}, \dots, x_{k+n}, y_{k+1}, \dots, y_{k+n}\} \setminus \{x_0, \dots, x_k, y_0, \dots, y_k\}) = z$.

Then

$$\tau Q \circ f_n^\omega(x) = Ff_n \circ \tau Q(x) = Ff_n \circ \tau Q(y) = \tau Q \circ f_n^\omega(y).$$

Let

$$x' = \lim_{n\to\infty} \{f_n^\omega(x)\},\ y' = \lim_{n\to\infty} \{f_n^\omega(y)\}.$$

We see that $x' \neq y'$ and $\tau Q(') = \tau Q(y')$. But

$$\operatorname{supp}(x') \cup \operatorname{supp}(y') = \{x_0, \dots, x_k, y_0, \dots, y_k, z\}.$$

This contradicticts to the injectivity of $\tau|\{x_0, \dots, x_k, y_0, \dots, y_k, z\}$. $\square$

Corollary 2.9.8. *Every finitely multiplicative normal functor is isomorphic to a power functor.*

By $U\colon \mathbf{Cgrp} \to \mathbf{Comp}$ we denote the forgetful functor from the category of compact Hausdorff topological groups $\mathbf{Cgrp}$ and their continuous homomorphisms.

Definition 2.9.9. We say that an endofunctor F in $\mathbf{Comp}$ can be *lifted* onto $\mathbf{Cgrp}$ if there exists an endofunctor $\bar{F}$ in $\mathbf{Cgrp}$ (a *lifting* of F) such that $U\bar{F} = FU$.

Theorem 2.9.10. *A normal functor F can be lifted onto* $\mathbf{Cgrp}$ *if and only if F is isomorphic to a power functor.*

Proof. Sufficiency is obvious. For necessity use Corollary 2.9.8. Let T be the topological group of complex numbers z with $|z| = 1$, e the identity of group T^ω. Supposing that F has a lifting F' onto $\mathbf{Cgrp}$, consider the homomorphism $f = (F'\operatorname{pr}_1, F'\operatorname{pr}_2)\colon F'(T^\omega \times T^\omega) \to F'(T^\omega) \times F'(T^\omega)$. Obtain

$$\begin{aligned}\ker(h) &= F'\operatorname{pr}_1^{-1}(e) \cap F'\operatorname{pr}_2^{-1}(e) = F(\{e\} \times T^\omega) \cap F'(T^\omega \times \{e\}) = \\ &= F'((\{e\} \times T^\omega) \cap (T^\omega \times \{e\})) = F'(\{(e,e)\}) = \{(e,e)\}.\end{aligned}$$

Since the map $F'\operatorname{pr}_2 | \ker(F'\operatorname{pr}_1)$ is an isomorphism in $\mathbf{Cgrp}$, we see that $\operatorname{im} h = F'(T^\omega) \times F'(T^\omega)$ and h is an isomorphism of topological groups. In particular, h is a homeomorphism.

Let X_1, X_2 be compact metrizable spaces. Supposing that X_1 and X_2 embed into T^ω by maps j_1, j_2 respectively, consider the following commutative diagram:

$$\begin{tikzcd} F(X_1 \times X_2) \arrow[r, "(F\operatorname{pr}_1, F\operatorname{pr}_2)"] \arrow[d, "F(j_1 \times j_2)"'] & FX_1 \times FX_2 \arrow[d, "Fj_1 \times Fj_2"] \\ F(T^\omega \times T^\omega) \arrow[r, "(F\operatorname{pr}_1, F\operatorname{pr}_2)"] & F(T^\omega) \times F(T^\omega). \end{tikzcd}$$

Since F preserves preimages and intersections, this diagram is a universal square. Therefore, $(F\,\mathrm{pr}_1, F\,\mathrm{pr}_2)\colon F(X_1 \times X_2) \to FX_1 \times FX_2$ is a homeomorphism and the functor F is multiplicative. By Corollary 2.9.8, it is isomorphic to a power functor. □

Theorem 2.9.11. *A normal functor F has a left adjoint functor if and only if F is a power functor.*

Proof. If F has a a left adjoint functor then F preserves limits (see, e. g., M. Barr, C. Wells [1985]) and, in particular, F is multiplicative. By Theorem 2.9.7, F is isomorphic to $(-)^\alpha$, $1 \le \alpha \le \omega$. On the other hand, the functor $(-)^\alpha$ has a left adjoint functor, namely, the functor $(-) \times \beta\alpha$. □

Problem

1. A topological semigroup S is called an *inverse semigroup* if for every $x \in S$ there exists a unique element $x^{-1} \in S$ satisfying the conditions $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$ and the map $x \mapsto x^{-1}\colon S \to S$ is continuous. The compact Hausdorff inverse semigroups and their continuous homomorphisms form the category **CISG**. Suppose that a normal functor F has a lifting to **CISG**. Is F a power functor?

2.10. Open and bicommutative functors. Characterization of G-symmetric power functors

A commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{i} & T \end{array} \tag{2.9}$$

is called *bicommutative* if the natural map

$$\chi\colon X \to Z \times_T Y = \{(z, y) \in Z \times Y \mid i(x) = h(y)\}, \quad \chi(x) = (g(x), f(x)),$$

$x \in X$, is onto. The map χ is called the *characteristic map* of the diagram (2.9). A functor is called *(finitely) bicommutative* if it preserves the bicommutative diagrams (consisting of finite spaces).

Proposition 2.10.1. *Let F be a normal functor with finite supports. Then the following conditions are equivalent:*
1) F is bicommutative;
2) F is finitely bicommutative.

Proof. It is necessary to prove that 2) implies 1). Given a bicommutative diagram (2.9), consider $a \in FZ$, $b \in FY$ such that $Fi(a) = Fh(b) \doteq c \in FT$. Then

$$L = \mathrm{supp}(a) \times_{\mathrm{supp}(c)} \mathrm{supp}(b) \subset Z \times_T Y$$

and, by bicommutativity of (2.9), there exists a finite subset $K \subset X$ such that the diagram

$$\mathcal{D} = \begin{array}{ccc} K & \xrightarrow{f|K} & \mathrm{supp}(b) \\ {\scriptstyle g|K}\downarrow & & \downarrow{\scriptstyle h|\mathrm{supp}(b)} \\ \mathrm{supp}(a) & \xrightarrow{i|\mathrm{supp}(a)} & \mathrm{supp}(c) \end{array}$$

is bicommutative (take, e.g., $K = \chi(L)$). By the finite bicommutativity of F, the diagram $F(\mathcal{D})$ is also bicommutative and, therefore, there exists $d \in FK$ such that $Fg(d) = a$ and $Ff(d) = b$. □

Example. There exists a finitely bicommutative functor which is not bicommutative.

For every $X \in |\mathbf{Comp}|$ denote by $\mathcal{R}_X$ the equivalence relation on $\exp X \times \exp X$ defined by the condition:

$(A, B)\mathcal{P}_X(A', B')$ if and only if either $A = A'$, $B = B'$ or $A = A' \in \exp^c X$ and $A \supset B$, $A' \supset B'$.

Let $FX = (\exp X \times \exp X)/\mathcal{R}_X$ and denote by $gX\colon \exp X \times \exp X \to FX$ the quotient map. For every map $f\colon X \to Y$ in **Comp** the map $Ff\colon FX \to FY$ can be found from the commutative diagram

$$\begin{array}{ccc} \exp X \times \exp X & \xrightarrow{\exp f \times \exp f} & \exp Y \times \exp Y \\ {\scriptstyle qX}\downarrow & & \downarrow{\scriptstyle qY} \\ FX & \xrightarrow{Ff} & FY \end{array}$$

It is easy to verify that F is a normal functor in **Comp** and

$$F|\mathbf{Comp}_0 \cong (\exp \times \exp)|\mathbf{Comp}_0.$$

Thus, F is a finitely bicommutative functor, because so is $\exp \times \exp$.

Show that F is not bicommutative. Let $\mathcal{D}$ denote the bicommutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "h"'] & 2^\omega \arrow[d, "f"] \\ 2^\omega \arrow[r, "f"] & I \end{tikzcd}$$

in which f is an onto map and $Z \in |\mathbf{Comp}_0|$. Fix $x, y \in I$, $x \neq y$. Then

$$qI(I, \{x\}) = qI(I, \{y\}) \in FI$$

and hence

$$Ff \circ q2^\omega(2^\omega, \{x'\}) = Ff \circ q2^\omega(2^\omega, \{y'\}) = qI(I, \{x\}),$$

whenever $f(x') = x$, $f(y') = y$. Evidently, there is no element $a \in FZ$ for which $Fg(a) = q2^\omega(2^\omega, \{x'\})$, $Fh(a) = q2^\omega(2^\omega, \{y'\})$, and therefore the diagram $F(\mathcal{D})$ is not bicommutative. This means that F is not bicommutative.

Theorem 2.10.2. *Let $f \colon X \to Y$ be an open map of limit spaces of compact σ-systems $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}, \mathcal{A}\}$ and $\mathcal{T} = \{T_\alpha, q_{\alpha\beta}, \mathcal{A}\}$. Then the set of indices $\alpha \in \mathcal{A}$ with $f \circ p_\alpha^{-1} = q_\alpha^{-1} \circ f_\alpha$ (for some $f_\alpha \colon X_\alpha \to Y_\alpha$) is cofinal in $\mathcal{A}$.*

For every compact Hausdorff space X denote by $\pi^3 X$ the following diagram:

$$\begin{tikzcd} X \times X \times X \arrow[r, "\pi_{12}"] \arrow[d, "\pi_{13}"'] & X \times X \arrow[d, "\pi_1"] \\ X \times X \arrow[r, "\pi_1"] & X \end{tikzcd}$$

and by $\pi^2 X$ the diagram

$$\begin{tikzcd} X \times X \times X \times X \arrow[r, "\pi_{12}"] \arrow[d, "\pi_{13}"'] & X \times X \arrow[d, "\pi_1"] \\ X \times X \arrow[r, "\pi_1"] & X \end{tikzcd}$$

(here π_i, π_{ij} are the projections onto the corresponding factors).

Proposition 2.10.3. *Let F be a normal functor. The bicommutativity of the diagram $F(\pi^3 Q)$ implies the bicommutativity of F.*

Proof. Denote by $\pi_{123}\colon Q^4 \to Q^3$, $\pi_{12}\colon Q^4 \to Q^2$, $\pi_{13}\colon Q^4 \to Q^2$, $\pi_1\colon Q^4 \to Q$ the projections of Q^4 onto the corresponding subproducts. Let $X \subset Q^4$ be a closed subset. Set

$$X_3 = \pi_{123}(X),\ X_{12} = \pi_{12}(X),\ X_{13} = \pi_{13}(X),\ X_1 = \pi_1(X).$$

Denote by D_X the following diagram

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d] & X_{12} \arrow[d] \\ X_{13} \arrow[r] & X_1 \end{tikzcd}$$

(the maps here are the restrictions of corresponding projections). Bicommutativity of D_X is equivalent to the following relation:

$$X_3 = \bigcap_{i=2,3} \pi_{123} \circ \pi_{1i}^{-1}(X_{1i}).$$

Let $x_{1i} \in Fx_{1i}$, $i = 2,3$, be points whose projections onto FX_1 in the diagram $F(D_X)$ are equal. By the bicommutativity of $F(\pi_3 Q)$, there exists $x \in F(Q^3)$ with $F(\pi_{1i} \circ \pi_{123}^{-1})(x_3) = x_{1i}$. Then

$$\mathrm{supp}(x_3) \subset \pi_{123} \circ \pi_{12}^{-1}(X_{12}) \cap \pi_{123} \circ \pi_{13}^{-1}(X_{13}) = X_3.$$

Thus, $x_3 \in FX_3$. Since the functor F is epimorphic, there exists $\overline{x} \in FX$ such that $F\pi_{123}(\overline{x}) = x_{123}$. Then $F\pi_{1i}(\overline{x}) = x_{1i}$, $i = 2,3$, and $F(D_X)$ is bicommutative. Since every bicommutative diagram of compact metrizable spaces can be represented as a diagram D_X for some X, we obtain the bicommutativity of F for compact metrizable spaces. The continuity of F implies the general case. $\square$

Let F be a normal functor and X' a closed subset of $X \in |\mathbf{Comp}|$. A map $r\colon X \to FX'$ is called an *F-valued retraction* of X onto X' if $r|X' = \eta X'$. Given additionally a map $f\colon X \to Y$, we say that r is *F-invariant* if $Ff \circ r = \eta Y \circ f$.

Proposition 2.10.4. *Let $f: X \to Y$ be an open map and X' is a closed subset of X which is an F-valued retract of X under an F-invariant retraction. Then the restriction $f|X'\colon X' \to FX'$ is open.*

Proof. Restricting, if necessary, f onto $f^{-1}(f(X'))$ we may assume that $f(X') = Y$. The composition of the maps

$$Y \xrightarrow{f^{-1}} \exp X \xrightarrow{\exp r} \exp FX' \xrightarrow{\exp \operatorname{supp}} \exp^2 X' \xrightarrow{uX'} X',$$

is lower-semicontinuous, because so are all the involved maps. Since F preserves preimages, this composition coincides with the multivalued map $f^{-1}|X'$. By Proposition 2.1.4, f is open. □

Corollary 2.10.5. *Let F be a normal functor. The openness of any map Ff implies the openness of f.*

Proof. Given a map $f\colon X \to Y$, we can consider the identity map 1_{FX} as an f-invariant F-valued retraction of FX onto X. □

Proposition 2.10.6. *Let F be a (weakly, almost) normal functor and $f\colon X \to Y$ a morphism in* **Comp**. *If the map $Ff|(Ff)^{-1}(\eta Y(Y))$ is open, then so is f.*

Proof. Suppose f is not open. Then there exist $x \in X$, $y \in Y$ and a net (y_α) in Y converging to y such that there is no net (x_α) in X converging to x and such that $f(x_\alpha) = y_\alpha$. Then the net $(\eta Y(y_\alpha))$ converges to ηY and it easily follows from the lower semicontinuity of supports that $\eta X(x)$ is not the limit of any net (a_α) in FX with $Ff(a_\alpha) = \eta Y(y_\alpha)$. □

Definition 2.10.7. An endofunctor F in **Comp** is called *open* if it preserves openness of open surjective maps.

Lemma 2.10.8. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ Y_2 & \xrightarrow[g_2]{} & Z \end{array}$$

be a bicommutative diagram, where all maps are onto. If f_2 is open, then so is g_1.

Proof. Let $U \subset Y_1$ be an open subset. Then $f_2 f_1^{-1}(U)$ is an open subset of Y_2. Since the diagram is bicommutative, $f_2 f_1^{-1}(U) = g_2^{-1}(V)$, for some $V \subset Z$. By closedness of the map g_2, we see that V is open. Since $V = g_1(U)$, we are done. □

Proposition 2.10.9. *Let $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a morphism of σ-systems $\{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ and $\{Y_\alpha, q_{\alpha\beta}; \mathcal{A}\}$. If all f_α are open and the morphism $(f_\alpha)_{\alpha\in\mathcal{A}}$ is bicommutative (in the sense that for every $\beta \geq \alpha$ the diagram*

$$\begin{tikzcd}
X_\beta \arrow[r, "f_\beta"] \arrow[d, "p_{\beta\alpha}"'] & Y_\beta \arrow[d, "q_{\beta\alpha}"] \\
X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha
\end{tikzcd}$$

is bicommutative) then the limit map $f = \varprojlim(f_\alpha)_{\alpha\in\mathcal{A}}$ is open.

Proof. Clearly, the limit map f satisfies the condition $fp_\alpha^{-1} = q_\alpha^{-1} f_\alpha$, for every $\alpha \in \mathcal{A}$. Given an open in X set of the form $p_\alpha^{-1}(U)$, for some open in X_α subset U, we see that the set $fp_\alpha^{-1}(U) = q_\alpha^{-1} f_\alpha(U)$ is open. Since the sets $p_\alpha^{-1}(U)$ form a base in X, we are done. □

A functor F is called *finitely open* if the map Ff is open for every surjective map f of finite spaces.

Proposition 2.10.10. *Let F be a normal bicommutative functor and the map $F\,\mathrm{pr}_1 \colon F(2^\omega \times 2^\omega) \to F(2^\omega)$ is open. Then F is an open functor.*

Proof. Let $f \colon X \to Y$ be an open surjective map of compact metrizable spaces. Without las of generality, we may suppose that $X \subset Y \times Q$ and $f = \mathrm{pr}_1 | X \colon X \to Y$. There exist surjective maps $g_1 \colon 2^\omega \to Y$ and $g_2 \colon 2^\omega \to Q$. Since every open map of compact metrizable spaces is 0-soft, there exist a map $\bar{g} \colon 2^\omega \times 2^\omega \to X$ such that the diagram

$$\begin{tikzcd}
& & Y \times Q \arrow[ddl, "pr_2"] \\
2^\omega \times 2^\omega \arrow[r, "\bar{g}"'] \arrow[urr, "g_1 \times g_2"] \arrow[d, "pr_1"'] & X \arrow[ur, hook] \arrow[d, "f"] & \\
2^\omega \arrow[r, "g_1"'] & Y &
\end{tikzcd} \tag{2.10}$$

is commutative and $\bar{g}|(g_1 \times g_2)^{-1}(X) = (g_1 \times g_2)|(g_1 \times g_2)^{-1}(X)$.

The square in diagram (2.10) is a bicommutative diagram. Applying F to this square, we also obtain a bicommutative diagram. Since the

map $F\,\mathrm{pr}_1 : F(2^\omega \times 2^\omega) \to f(2^\omega)$ is open, we conclude, by Lemma 2.10.8, that so is the map Ff.

Now suppose that $f: X \to Y$ is an open map of compact Hausdorff spaces. By Theorem 1.1.2, f can be represented as the limit of a bicommutative morphism (f_α) of σ-systems. By Lemma 2.10.8, all f_α are open, and hence so are Ff_α. By Proposition 2.10.9, Ff is open. □

The following propositions give examples of open functors.

Proposition 2.10.11. *The functor* exp *is open.*

Proof. This is a consequence of the equality

$$\exp f\langle U_1, \dots, U_n\rangle = \langle f(U_1), \dots, f(U_n)\rangle$$

for a map $f: X \to Y$. □

Proposition 2.10.12. *The functor* $\exp^c$ *is not open.*

Proof. Let $\mathrm{pr}_1 : S^1 \times [0,1] \to S^1$ denote the projection, $S^1 = \{(x,y) \mid x^2+y^2 = 1\}$. Let $A = (\cos 3\pi t, \sin 3\pi t, t) \in \exp^c(S^1 \times [0,1])$, then $\exp^c \mathrm{pr}_1(A) = S^1$ and it is easy to see that $\exp^c \mathrm{pr}_1(A') = S^1$ for any A' sufficiently close to A. Thus, the map $\exp^c \mathrm{pr}_1$ is not open. □

Proposition 2.10.13. *The functors G, λ, N are open.*

Proof. For the functor G this follows from the equalities $Gf(U^\pm) = (f(U))^\pm$ (for a map $f: X \to Y$ and an open subset U of X).

Prove that $\lambda f(U^+) = (f(U))^+$. For this, it is sufficient to prove the inclusion $\supset$. Let $\mathcal{M} \in (f(U))^+$, then there exists $M \in \mathcal{M}$, $M \subset f(U)$ and, therefore, there exists a set $N \in \exp X$ such that $N \subset U$. The linked system $\mathcal{N}' = \{N\} \cup \{f^{-1}(A) \mid A \in \mathcal{M}\}$ is contained, by Zorn lemma, in a maximal linked system $\mathcal{N}$. Obviously, $\lambda f(\mathcal{N}) = \mathcal{M}$ and $\mathcal{N} = U^+$. This proves openness of λ. The case of functor N can be considered similarly. □

Proposition 2.10.14. *The functors N_k, $k \geq 4$, are not open.*

Proof. Let

$$X = \{x, y, z\} \cup (\bigcup \{\{x'_n, x''_n, y_n, z_n\} \mid n \in \mathbb{N}\},$$
$$Y = \{a, b\} \cup (\bigcup \{\{a'_n, a''_n, b_n\} \mid n \in \mathbb{N}\},$$

where all the points $x, y, z, x'_n, x''_n, y_n, z_n, a, b, a'_n, a''_n, b_n$ are distinct and the compact metrizable topologies on X and Y respectively are given by the following conditions:

1) the points $x'_n, x''_n, y_n, z_n, a, b, a'_n, a''_n, b_n$ are isolated;

2) the sequences $\{x'_n \mid n \in \mathbb{N}\}$, $\{x''_n \mid n \in \mathbb{N}\}$, $\{y_n \mid n \in \mathbb{N}\}$, $\{z_n \mid n \in \mathbb{N}\}$, $\{a'_n \mid n \in \mathbb{N}\}$, $\{a''_n \mid n \in \mathbb{N}\}$, $\{b_n \mid n \in \mathbb{N}\}$ converge respectively to the points x, x, y, z, a, a, b.

Define a map $f\colon X \to Y$ by the formulae: $f(x) = a$, $f(y) = f(z) = b$, $f(x'_n) = a'_n$, $f(x''_n) = a''_n$, $f(y_n) = f(z_n) = b_n$.

Let

$$\begin{aligned}\mathcal{A} =&\{A \in \exp X \mid A \supset \{x, y\}\} \cup \{A \in \exp X \mid A \supset \{x, z\}\} \in N_k X,\\ \mathcal{B} =&\{B \in \exp Y \mid B \supset \{a'_n, b_n\}\} \cup \{B \in \exp X \mid B \supset \{a''_n, b_n\}\} \in N_k Y.\end{aligned}$$

Then $N_k f(\mathcal{A}) = \mathcal{B} = \lim_{n\to\infty} \mathcal{B}_n$. Assuming that the map $N_k f$ is open, we obtain that there exists a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ in $N_k X$ such that $\lim_{n\to\infty} \mathcal{A}_n = \mathcal{A}$ and $N_k f(\mathcal{A}_n) = \mathcal{B}_n$.

The sets

$$\begin{aligned}U =&\{x\} \cup \{x'_n \mid n \in \mathbb{N}\} \cup \{x''_n \mid n \in \mathbb{N}\},\\ V =&\{y\} \cup \{y'_n \mid n \in \mathbb{N}\},\ W = \{z\} \cup \{z'_n \mid n \in \mathbb{N}\}\end{aligned}$$

are open in X. Let $D = (U \cup V)^+ \cap (U \cup W)^+$, then for some $n_0 \in \mathbb{N}$ we have $\mathcal{A}_{n_0} \in D$. There exist $A', A'' \in \mathcal{A}_{n_0}$ such that $A' \subset U \cup V$, $A'' \subset U \cup W$. Besides, there exist $C', C'' \in \mathcal{A}_{n_0}$ such that $f(C') = \{b_{n_0}, a'_{n_0}\}$, $f(C'') = \{b_{n_0}, a''_{n_0}\}$. But then $A' \cap A'' \cap C' \cap C'' = \varnothing$ and we obtain a contradiction.

$\square$

Proposition 2.10.15. *The functor P is open.*

Proof. Let $f\colon X \to Y$ be an open map in **Comp**. By Proposition 2.1.4, it is sufficient to prove that the map $(Pf)^{-1}\colon PY \to \exp PY$ is continuous. Assuming the contrary, we see that there exists $\nu_0 \in PY$ and a net (ν_α) in $P_\omega Y \subset PY$ converging to ν_0 and such that the sets $A_\alpha = (Pf)^{-1}(\nu_\alpha)$ fail to converge to $(Pf)^{-1}(\nu_0)$. Without loss of generality, we may assume that $\lim_\alpha A_\alpha = A$, where $A \subset (Pf)^{-1}(\nu_0)$ and there exists $\mu_0 \in (Pf)^{-1}(\nu_0) \setminus A$.

Since all A_α are convex in PX, A is convex as well. The Hahn-Banach theorem implies that there exist $\varphi \in C(X)$ and $\varepsilon > 0$ such that

$\mu_0(\varphi) \geq \varepsilon + \mu(\varphi)$ for every $\mu \in A$. Define the function $\varphi^*\colon Y \to \mathbb{R}$ by the formula $\varphi^*(y) = \max\{\varphi(x) \mid x \in f^{-1}(y),\ y \in Y\}$. Continuity of φ^* is a consequence of openness of f.

Let

$$\nu_\alpha = \sum_{k=1}^{n_\alpha} \beta_{\alpha,k}\delta_{y_{\alpha,k}},\ y_{\alpha,k} \in Y,\ \beta_{\alpha,k} \in [0,1].$$

There exist points $x_{\alpha,k} \in X$, for which $\varphi(x_{\alpha,k}) = \varphi^*(y_{\alpha,k})$. Let $\mu_\alpha = \sum_{k=1}^{n_\alpha} \beta_{\alpha,k}\delta_{x_{\alpha,k}}$ and let μ be a limit point of the net (μ_α). Then $\mu \in A$ and

$$\begin{aligned}\mu(\varphi) &= \lim_\alpha \mu_\alpha(\varphi) = \lim_\alpha \sum_{k=1}^{n_\alpha} \beta_{\alpha,k}\varphi(x_{\alpha,k})\\ &= \lim_\alpha \sum_{k=1}^{n_\alpha} \beta_{\alpha,k}\varphi^*(y_{\alpha,k}) = \lim_\alpha \nu_\alpha(\varphi^*)\\ &= \nu_0(\varphi^*) = \mu_0(\varphi^* \circ f) \geq \mu_0(\varphi).\end{aligned}$$

This gives a contradiction. □

A closed subset A of OX is called *O-convex* if for each $\mu \in OX$ with $\inf A \leq \mu \leq \sup A$ we have $\mu \in A$.

Lemma 2.10.16. *Let $f\colon X \to Y$ be a map in* **Comp**. *For each $\nu \in O(Y)$ the preimage $(Of)^{-1}(\nu)$ is an O-convex subset of $O(X)$.*

Proof. Put $A = (Of)^{-1}(\nu)$ and consider any $\mu \in A$. Let us show first that $\inf A, \sup A \in A$. Let $\varphi \in C(Y)$. Then $\sup A(\varphi \circ f) = \sup\{\mu(\varphi \circ f) \mid \mu \in A\} = \nu(\varphi)$. Hence $\sup A \in A$. The similar arguments yield $\inf A \in A$.

Now, if $\inf A \leq \mu \leq \sup A$ for some $\mu \in OX$ then for each $\varphi \in C(Y)$ we have $\nu(\varphi) = \inf A(\varphi \circ f) \leq \mu(\varphi \circ f) \leq \sup(\varphi \circ f) = \nu(\varphi)$. Hence $\mu \in A$. □

Proposition 2.10.17. *The functor O is open.*

Proof. Now let a map $f\colon X \to Y$ be open. Suppose that Of is not open. Then there exists a functional $\mu_0 \in OX$, a net of functionals $(\nu_\alpha)_{\alpha\in\mathcal{A}}$ in OY converging to $\nu_0 = Of(\mu_0)$ and a neighbourhood W of μ_0 such that $(Of)^{-1}(\nu_\alpha) \cap W = \varnothing$ for each $\alpha \in \mathcal{A}$. Since $O_\omega Y$ is a dense subset

of OY, we can suppose that all ν_α are supported on finite sets. Since OX is compact, we can assume that the net $(A_\alpha = (Of)^{-1}(\nu_\alpha))_{\alpha\in\mathcal{A}}$ converges in $\exp OX$ to some closed subset $A \subset OX$. It is easy to check that $A \subset (Of)^{-1}(\nu_0)$ and $\mu_0 \notin A$.

By Lemma 2.10.16 all the sets A_α are O-convex. It is easy to see that A is O-convex as well. Put $\alpha_1 = \inf A$ and $\alpha_2 = \sup A$. Since $\mu_0 \notin A$, there exists $\varphi \in C(X)$ with $\mu_0(\varphi) > \alpha_2(\varphi)$ or $\mu_0(\varphi) < \alpha_1(\varphi)$. For each $\alpha \in \mathcal{A}$ we have $\nu_\alpha O\{y_{\alpha 1}, \dots, y_{\alpha n_\alpha}\} \subset O(Y)$. Choose for each $y_{\alpha i}$ a point $x_{\alpha i}$ such that $f(x_{\alpha i}) = y_{\alpha i}$ and $\varphi(x_{\alpha i}) = \varphi^*(y_{\alpha i})$. Define the embedding $j_\alpha\colon \{y_{\alpha 1}, \dots, y_{\alpha n_\alpha}\} \to X$ by the formula $j_\alpha(y_{\alpha i}) = x_{\alpha i}$ and put $\mu_\alpha = O(j_\alpha)\nu_\alpha$. Let μ be a limit point of the net μ_α, then $\mu \in A$. Since $\mu_\alpha(\varphi) = \nu_\alpha(\varphi^*)$, we have $\mu(\varphi) = \nu_0(\varphi^* \circ f) \geq \mu_0(\varphi)$. This gives a contradiction. □

Proposition 2.10.18. *Suppose that $f\colon X \to Y$ is an open map in* **Comp**. *If $|f^{-1}(y)| \geq n-1$ for some $y \in Y$, then the map $\lambda_n f$ is not open, $n \geq 3$.*

Proof. Suppose that $z \in Y$, $z \neq y$. There exists $A \subset f^{-1}(y)$, $|A| = n-1$, and $x \in f^{-1}(x)$. Let $\mathcal{M} \in \lambda_n X$ be such that $\mathcal{M} \supset \{A\} \cap \{\{a,x\} | a \in A\}$. Since y is nonisolated, there exists a sequence $(\mathcal{N}_i)_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \mathcal{N}_i = \nu X(y)$, $\lim_{i\to\infty} \operatorname{supp}(\mathcal{N}_i) = \{y\}$ (in $\exp Y$), and $|\operatorname{supp}(\mathcal{N}_i)| = n$ for every i. Obviously, there is no sequence $(\mathcal{M}_i)_{i=1}^{\infty}$ in $\lambda_n X$ such that $\lambda_n f(\mathcal{M}_i) = \mathcal{N}_i$ and $\lim_{i\to\infty} \mathcal{M}_i = \mathcal{M}$. □

Corollary 2.10.19. *The functors λ_n, $n \geq 3$, are not open.*

Proposition 2.10.20. *Every open normal functor is bicommutative.*

Proof. Let $\mathcal{S}$ be an inverse σ-system with the limit Q^{ω_1} formed by the projections of Q^{ω_1} onto the countable powers. Then the natural morphism of $F(\mathcal{S}\times\mathcal{S})$ into $F(\mathcal{S})$ has the limit morphism $F\operatorname{pr}_1\colon F(Q^{\omega_1} \times Q^{\omega_1}) \to FQ^{\omega_1}$. By Theorem 2.10.2, the openness of $F\operatorname{pr}_1$ implies that this morphism contains a bicommutative submorphism. All square diagrams of this submorphism, which are parallel to the limit map, are homeomorphic to $F(\pi^2 Q)$. Thus, the diagram $F(\pi^2 Q)$ is bicommutative. It is easy to prove that then the diagram $F(\pi^3 Q)$ is also bicommutative. Now it remains to use Proposition 2.10.3. □

Note that the finite bicommutativity is an algebraic property of functors.

2.10.1. Characterization of G-symmetric power functors

A natural transformation $\varphi = (\varphi X)\colon F \to G$ is called *open* if so are all its components $\varphi X\colon FX \to GX$. The main result of this section is the following

Theorem 2.10.21. *For a normal functor F of finite degree $n \geq 1$ the following conditions are equivalent:*
1) F is open;
2) F is bicommutative;
3) there exists an open natural transformation $\varphi\colon (-)^n \to F$;
4) $F \cong SP^n_G$ for some subgroup $G \subset S_n$.

Proof. Proposition 2.10.20 implies 1)$\Rightarrow$2). Since the map $\pi_G X\colon X^n \to SP^n_G$, $\pi_G X(x_0, \dots, x_{n-1}) = [x_0, \dots, x_{n-1}]_G$, is open, we have 4)$\Rightarrow$1).

2)$\Rightarrow$3). Fix $a_0 \in Fn$, $\deg(a_0) = n$. Identify the spaces X^n and $C(n, X)$ for every X. Define a map $\xi X\colon X^n \to FX$ by the formula $\xi X(f) = Ff(a_0)$, $f \in C(n, X)$. The map ξX is continuous as a restriction of the Basmanov map. Clearly, $\xi = (\xi X)\colon (-)^n \to F$ is a natural transformation.

Show that ξX is surjective. Let $b \in FX$. Using the bicommutativity of F, choose $c \in F(n \times X)$ such that $F\,\mathrm{pr}_1(c) = a_0$, $F\,\mathrm{pr}_2(c) = b$. Then the set $\mathrm{supp}(c) \subset n \times n$ is a graph of map $g\colon n \to X$ with $Fg(a_0) = b$. Hence, $b \in \xi X(X^n)$.

Lemma 2.10.22. *Let $\xi\colon (-)^n \to F$ be a surjective natural transformation and* $\deg F = n$. *Then* $\mathrm{supp}\,\xi X(a) = \mathrm{supp}(a)$ *for every $a \in X^n$.*

Proof. Let $b \in Fn$, $\deg(b) = n$, and a point $c \in n^n$ be such that $\xi n(c) = b$. Then there exists $f \in C(n, X)$ such that $a = f^n(c)$. But then $\mathrm{supp}(a) = f(n)$ and, therefore, $\mathrm{supp}\,\xi X(a) = \mathrm{supp}(\xi X \circ f^n(c)) = \mathrm{supp}(Ff \circ \xi n(c)) = f(\mathrm{supp}(b)) = f(n) = \mathrm{supp}(a)$. □

Lemma 2.10.23. *If for every $f \in C(n, n)$ the following diagram*

$$\begin{tikzcd} n^n \arrow[r, "f^n"] \arrow[d, "\xi n"'] & n^n \arrow[d, "\xi n"] \\ Fn \arrow[r, "Ff"] & Fn \end{tikzcd} \tag{2.11}$$

is bicommutative then ξ is an open natural transformation.

Proof. Since $\deg F = n$, by the bicommutativity of diagram (2.11) for every $f \in C(n,n)$, the following commutes for every $g\colon X \to Y$:

$$\begin{array}{ccc} X^n & \xrightarrow{g^n} & Y^n \\ \downarrow{\scriptstyle \xi X} & & \downarrow{\scriptstyle \xi Y} \\ FX & \xrightarrow{Fg} & FY. \end{array}$$

Let X be a compact metrizable space, $a \in FX$ a non-isolated point, $\{b_i\} \subset FX \setminus \{a\}$ a converging to a sequence, and $c \in (\xi X)^{-1}(a)$. Let $\deg(a) = l$, $\operatorname{supp}(a) = \{x_0, \ldots, x_{l-1}\}$. Consider mutually disjoint neighborhoods $U_0, \ldots, U_{l-1}$ of $x_0, \ldots, x_{l-1}$, respectively.

By the bicommutativity of (2.11) for a constant map $f\colon n \to n$ we have that ξn is surjective and hence, F is a finite functor. By Corollary 2.4.6 F has continuous supports. Thus, we can suppose that $\operatorname{supp}(b_i) \subset U_0 \cup \cdots \cup U_{l-1}$.

Let $h\colon U_0 \cup \cdots \cup U_{l-1} \to \{x_0, \ldots, x_{l-1}\}$ be a map such that $h(U_j) = \{x_j\}$, $j \in l$. Then

$$\lim_{i\to\infty} Fh(b_i) = Fh(\lim_{i\to\infty} b_i) = Fh(a) = a.$$

By the finiteness of F, we see that $Fh(b_i) = a$ for all $i \geq i_0$, $i_0 \in \mathbb{N}$. Using Lemma 2.10.22 and the bicommutativity of the previous diagram, choose a sequence $\{d_i\} \subset X^n$, $i \geq i_0$, such that $\xi X(d_i) = b_i$, $h^n(d_i) = c$.

Let $\lim\limits_{k\to\infty} d_{i_k} = c'$ for some sequence $\{i_k\}$. Then

$$h^n(c') = h^n(\lim_{k\to\infty} d_{i_k}) = c.$$

Since $\operatorname{supp}(c') = \{x_0, \ldots, x_{l-1}\}$ and the map $h|\{x_0, \ldots, x_{l-1}\}$ is the identity map, we have $c' = c$.

Hence, $\lim_{i\to\infty} d_i = c$. Therefore, the map $(\xi X)^{-1}\colon FX \to \exp(X^n)$ is lower semicontinuous and ξX is open.

By the continuity of F and $(-)^n$, we obtain the general case. □

Returning to the proof of Theorem 2.10.21, show the openness of ξX. By Lemma 2.10.23, for this it is sufficient to prove that diagram (2.11) is bicommutative.

Let $a, b \in Fn$, $c \in n^n$ be such that $Ff(b) = a = \xi n(c)$. At first, consider the case $\deg(b) = n$. There exist $d \in n^n$, $\deg(d) = n$, and

a map $g: n \to n$ such that $g^n(d) = c$. Put $b' = \xi n(d)$. Then $\deg(b') = n$ and $Fg(b') = a$. Consider a universal square of $f, g: n \to n$,

$$\begin{tikzcd} K \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & n \arrow[d, "f"] \\ n \arrow[r, "g"] & n. \end{tikzcd}$$

By the bicommutativity of F, there exists $b_1 \in FK$ with

$$F\pi_1(b_1) = b', \ F\pi_2(b_1) = b.$$

Then $\deg(b_1) = n$. Obviously, the composition

$$h = \big(\pi_2|\operatorname{supp}(b_1)\big) \circ \big(\pi_1|\operatorname{supp}(b_1)\big)^{-1} : n \to n$$

satisfies the following: $Fh(b') = b$ and $g = f \circ h$. Set $d' = h^n(d)$. Then

$$\xi n(d') = \xi n \circ h^n(d) = Fh \circ \xi n(d) = Fh(b') = b$$

and

$$f^n(d') = f^n \circ h^n(d) = g^n(d) = c.$$

Now consider the general case. Let $d_1 \in (\xi n)^{-1}(b)$. There exist $d_2 \in n^n$, $\deg(d_2) = n$, and $\alpha \in C(n, n)$ such that $\alpha^n(d_2) = d_1$. Let $b_2 = \xi n(d_2)$. Then $\deg(b_2) = n$, $F(f \circ \alpha)(b_2) = a$. By the above arguments, there exists $\bar{d} \in Fn$ such that $\xi n(\bar{d}) = b_2$ and $(f \circ \alpha)^n(\bar{d}) = c$. Putting $d' = \alpha^n(\bar{d})$, obtain that $\xi n(d') = b$ and $f^n(d') = c$, i.e., diagram (2.11) is bicommutative.

3)$\Rightarrow$4). Let $\xi: (-)^n \to F$ be an open natural transformation. Show that the map $\xi k: k^n \to Fk$, $k \le n$, is surjective.

Let $i: k \to I = [0, 1]$ be an embedding. It is easy to see that FI is a connected space. Thus, ξI is epi. Let $a \in I^n$ and $\xi I(a) = b \in Fk \subset FI$. Denoting by r a retraction of $\operatorname{supp}(a)$ onto $\operatorname{supp}(b)$, obtain $a' = r^n(a) \in k^n \subset I^n$ and $\xi k(a') = b$. Since $\deg F = n$, the map ξX is surjective for every X.

Fix $a_0 \in Fn$, $\deg(a_0) = n$. Put

$$G = \{\sigma \in S_n \mid F\sigma(a_0) = a_0\}.$$

Show that there exists $\xi'\colon SP^n_G \to F$ with $\xi = \xi' \circ \pi_G$. For this we only need to prove that for every $(x_0, \dots, x_{n-1}), (y_0, \dots, y_{n-1}) \in X^n$ with $[x_0, \dots, x_{n-1}]_G = [y_0, \dots, y_{n-1}]_G$ the equality

$$\xi X(x_0, \dots, x_{n-1}) = \xi X(y_0, \dots, y_{n-1})$$

holds.

There exists $c \in n^n$ such that $\xi n(c) = a_0$. Then $\deg(c) = n$ and there exists a map $f\colon n \to X$ such that $f^n(c) = (x_0, \dots, x_{n-1})$. For some $\sigma \in G$ we have $f^n \circ \sigma^n(c) = (y_0, \dots, y_{n-1})$. Thus,

$$\begin{aligned}\xi X(y_0, \dots, y_{n-1}) &= \xi X \circ f^n \circ \sigma^n(c) = Ff \circ F\sigma \circ \xi n(c) = \\ &= Ff \circ F\sigma(a_0) = Ff(a) = Ff \circ \xi n(c) = \\ &= \xi X(x_0, \dots, x_{n-1}).\end{aligned}$$

Hence, we can define a map $\xi' X\colon SP^n_G X \to FX$ by the formula:

$$\xi' X([x_0, \dots, x_{n-1}]_G) = \xi X(x_0, \dots, x_{n-1}).$$

Since $\pi_G X$ is open and epi, $\xi' X$ is continuous. It is easy to see that the family $(\xi' X)_{X \in \mathbf{Comp}}$ forms a natural transformation $\xi'\colon SP^n_G \to F$.

By definition, $\xi' X$ is one-to-one on the set of points of degree $\leq n$. Moreover, it is open as an open divisor of an open map. Since the points of degree $\leq n$ in SP^n_G and FI form dense subsets, $\xi' I$ is a homeomorphism. Therefore, ξ is a functorial isomorphism. □

2.10.2. Strongly bicommutative and strongly open functors

Given a sequence of maps $f_i\colon X \to Y_i$, $i = 0, \dots, n$, we denote by $\otimes_{i=0}^n f_i$ the diagonal map $(f_i)_{i=0}^n$ considered as a map onto its image (i. e. $(f_i)_{i=0}^n = j \circ \otimes_{i=0}^n f_i$, where j denotes the inclusion of $(f_i)_{i=0}^n(X)$ into $Y_0 \times \dots \times Y_n$).

Definition 2.10.24. A functor F is called *strongly open* if for every sequence $f_i\colon X \to Y_i$, $i = 0, \dots, n$ of open onto maps in $\mathbf{Comp}$ the map $\otimes_{i=0}^n Ff_i$ is open.

This notion is of little interest in investigations of normal functors, because of the following fact.

Proposition 2.10.25. *No normal functor is strongly open.*

Proof. It is easy to construct two open maps $f_i\colon X \to Y_i$, $i = 0, 1$, for which the map $f_0 \otimes f_1$ is not open. Since any normal functor F preserves intersections and preimages, we see that for every $(y_0, y_1) \in (f_0 \otimes f_1)(X)$ the following holds:

$$\begin{aligned}
&(F(f_0 \otimes f_1))^{-1}(\eta(f_0 \otimes f_1)(X)(y_0, y_1)) \\
=&F((f_0 \otimes f_1)^{-1}(y_0, y_1)) = F(f_0^{-1}(y_0) \cap f_1^{-1}(y_1)) \\
=&(Ff_0)^{-1}(\eta Y_0(y_0)) \cap (Ff_1)^{-1}(\eta Y_1(y_1)) \\
=&((Ff_0) \otimes (Ff_1))^{-1}(\eta Y_0(y_0), \eta Y_1(y_1)).
\end{aligned}$$

Thus, by Proposition 2.10.6, the map $(Ff_0) \otimes (Ff_1)$ is not open. $\square$

However, in the class of weakly normal functors the situation is better.

Proposition 2.10.26. *The functor N is strongly open.*

Proof. Let $\mathcal{A} \in NX$ and

$$\bar{\mathcal{B}} = (\mathcal{B}_0, \ldots, \mathcal{B}_n) = (\otimes_{i=0}^{n} Nf_i)(\mathcal{A}) \in \prod_{i=0}^{n} Y_i.$$

Consider a neighborhood of $\mathcal{A}$ of the form $U_1^+ \cap \cdots \cap U_k^+ \cap V_1^- \cap \cdots \cap V_m^-$, where U_i and V_j are open in X. Then there exist $A_i \in \mathcal{A}$ such that $A_i \subset U_i$ and closed sets $B_i \subset V_i$ such that $B_i \cap A \neq \varnothing$ for every $A \in \mathcal{A}$. Consider neighborhoods U_i' of A_i and V_i' of B_i such that $\overline{U_i'} \subset U_i$ and $\overline{V_j'} \subset V_j$. Obviously, the set

$$\begin{aligned}
W = &\prod_{i=0}^{n}((f_i(U_1'))^+ \cap \cdots \cap (f_i(U_k')^+) \\
&\cap ((f_i(V_1'))^- \cap \cdots \cap (f_i(V_m')^-) \cap (\otimes_{i=0}^{n} Nf_i)(NX)
\end{aligned}$$

is a neighborhood of $\bar{\mathcal{B}}$ in $(\otimes_{i=0}^{n} Nf_i)(NX)$.

To finish the proof, we need only to show that

$$W \subset (\otimes_{i=0}^{n} Nf_i)(U_1^+ \cap \cdots \cap U_k^+ \cap V_1^- \cap \cdots \cap V_m^-).$$

Indeed, suppose that $\bar{\mathcal{C}} = (\mathcal{C}_0, \ldots, \mathcal{C}_n) \in W$. There exists $\mathcal{M} \in NX$ such that $(\otimes_{i=0}^n Nf_i)(\mathcal{M}) = \bar{\mathcal{C}}$. Thus, the system $\mathcal{A}_1 = \cup_{i=0}^n \{f_i^{-1}(C) \mid C \in \mathcal{C}_i\} \in \exp^2 X$ is contained in $\mathcal{M}$ and, therefore, linked. Put

$$\mathcal{D} = \{D \in \exp X \mid D \supset D' \in \mathcal{A}_1 \cup \{\overline{U_1'}, \ldots, \overline{U_k'}\}\}.$$

It is easy to see that $(\otimes_{i=0}^n Nf_i)(\mathcal{D}) = \bar{\mathcal{C}}$ and

$$\mathcal{D} \in U_1^+ \cap \cdots \cap U_k^+ \cap V_1^- \cap \cdots \cap V_m^-.$$

□

Definition 2.10.27. A functor F is called *strongly bicommutative* if for every finite collection of bicommutative diagrams

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "g"'] & Y_i \arrow[d, "h_i"] \\ Y \arrow[r, "k_i"'] & Z_i \end{tikzcd} \quad , \ i = 0, \ldots, n \tag{2.12}$$

the diagram

$$\begin{tikzcd} FX \arrow[r, "\otimes_{i=0}^n Ff_i"] \arrow[d, "Fg"'] & (\otimes_{i=0}^n Ff_i)(FX) \arrow[d, "\prod_{i=0}^n Fh_i|(\otimes_{i=0}^n Fh_i)(FX)"] \\ FY \arrow[r, "\otimes_{i=0}^n Fk_i"'] & (\otimes_{i=0}^n)(FY) \end{tikzcd}$$

is also bicommutative.

Proposition 2.10.28. *The functor N is strongly bicommutative.*

Proof. Given a sequence 2.12 of bicommutative diagrams in **Comp**, denote, for the sake of brevity, the map $\prod_{i=0}^n Nh_i|(\otimes_{i=0}^n Nh_i)(NX)$ by H. Suppose that $\bar{\mathcal{A}} = (\mathcal{A}_0, \ldots, \mathcal{A}_n) \in (\otimes_{i=0}^n Nf_i)(NX)$. We need to prove the equality

$$Ng \circ (\otimes_{i=0}^n Nf_i)^{-1}(\bar{\mathcal{A}}) = (\otimes_{i=0}^n Nk_i)^{-1} \circ H(\bar{\mathcal{A}}).$$

To this end, it is sufficient to verify that for every $\mathcal{B} \in NY$ such that $(\otimes_{i=0}^n Nk_i)(\mathcal{B}) = H(\mathcal{B})$ there exists $\mathcal{C} \in NX$ for which $Ng(\mathcal{C}) = \mathcal{B}$ and $(\otimes_{i=0}^n Nf_i)(\mathcal{C}) = \bar{\mathcal{A}}$.

Indeed, if $\mathcal{B}$ is as above, then the system

$$\mathcal{B}' = \cup_{i=0}^{n}\{k_i^{-1}(h_i(A)) \mid A \in \mathcal{A}_i\}$$

is contained in $\mathcal{B}$ and, therefore, linked. The system $\mathcal{C}' = \cup_{i=0}^{n}\{f_i^{-1}(A) \mid A \in \mathcal{A}_i\}$ is also linked, because $\bar{\mathcal{A}} \in (\otimes_{i=0}^{n} Nf_i)(NX)$. By bicommutativity of diagrams (2.12), we have

$$\exp^2 g(\mathcal{C}') = \cup_{i=0}^{n}\{g(f_i^{-1}(A)) \mid A \in \mathcal{A}_i\} = \mathcal{B}' \subset \mathcal{B}.$$

This implies that $\mathcal{C}'' = \mathcal{C}' \cup \{g^{-1}(B) \mid B \in \mathcal{B}\}$ is linked in X.

Finally, let $\mathcal{C} = \{C \in \exp X \mid C \supset C'' \in \mathcal{C}''\} \in NX$. It is easy to verify that $\mathcal{C}$ is as required. □

Exercise

1. Prove that the inclusion hyperspace functor G is strongly open and strongly bicommutative.

Problems

1. *(E. V. Shchepin)* Is every bicommutative normal functor open?
2. Is Theorem 2.10.21 still valid in the class of weakly normal functors?
3. Let F be a normal functor and X a compact metric space. An element $a \in FX$ is called *generating* if for every compact metric Y and $b \in FY$ there exists $f\colon X \to Y$ such that $b = Ff(a)$. Is the condition of existence of generating element equivalent to bicommutativity?

2.11. Characteristic of normal functors

Let

$$\mathcal{D} = \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{u} & T \end{array}$$

be a commutative diagram in **Comp**. Recall that

$$Y \times_T Z = \{(y, z) \in Y \times Z \mid g(y) = u(z)\}.$$

The natural map

$$\chi\colon X \to Y \times_T Z, \quad \chi(x) = (f(x), h(x)),$$

$x \in X$, is called the *characteristic map* of the diagram $\mathcal{D}$.

A number $n \in \omega$ is called a *characteristic number* of $\mathcal{D}$ if there exists a fiber $\chi^{-1}(a)$ of the map χ having cardinality n. The set of all characteristic numbers of $\mathcal{D}$ (supplemented with the symbol ∞ whenever the map χ has infinite fibers) is called the *characteristic* of $\mathcal{D}$.

For every $K \in |\mathbf{Comp}|$ by $\pi^3 K$ we denote the diagram

$$\begin{tikzcd} K \times K \times K \arrow[r, "\pi_{13}"] \arrow[d, "\pi_{12}"'] & K \times K \arrow[d, "\pi_1"] \\ K \times K \arrow[r, "\pi_1"] & K \end{tikzcd}$$

(π_i, π_{ij} are the projections; the indices are the numbers of factors onto which the projections act).

The *K-characteristic* $\chi_K(F)$ of a normal functor F is the characteristic of the diagram $F(\pi^3 K)$. We use the notation $\chi_0(F)$ if K is the Cantor set and $\chi(F)$ if K is the Hilbert cube Q.

Further, F is a normal functor in $\mathbf{Comp}$. Recall that a map $f\colon X \to Y$ is *irreducible* if there is no closed subset A of X such that $f(A) = Y$.

Proposition 2.11.1. *Suppose that $K \in |\mathbf{Comp}|$ has no isolated point. Then the characteristic map $F(\pi^3 K)$ is irreducible.*

Proof. Since K has no isolated point, the set

$$L_n = \{f \in C(n, K^3) \mid \pi_1 f \text{ is injective}\}$$

is dense in $C(n, K^3)$, for every n. Therefore, the set

$$\mathcal{A} = \{Ff(a) \mid f \in L_n,\ a \in Fn,\ n \in \mathbb{N}\}$$

is dense in $F(K^3)$.

Let $b \in \mathcal{A}$, $b = Ff(a)$, where $f \in L_n$ and $a \in Fn$. Suppose that $c \in F(K^3)$ and $\chi F(\pi^3 K)(c) = \chi F(\pi^3 K)(b)$, then $c \in FA$, where

$$A = \pi_{12}^{-1}(\pi_{12}(\mathrm{supp}(b))) \cap \pi_{13}^{-1}(\pi_{13}(\mathrm{supp}(b))).$$

Obviously, the restriction $\pi_1|A\colon A \to \pi_1(A)$ is a homeomorphism and

$$c = F((\pi_1|A)^{-1})(F(\pi_1|A)(c) = F((\pi_1|A)^{-1})(F(\pi_1|A)(b) = b.$$

We have proved that all points in $\mathcal{A}$ are non-multiple points of the map $F(\pi^3 K)$. Since $\mathcal{A}$ is dense in $F(K^3)$, this map is irreducible. $\square$

Proposition 2.11.2. *If $K_1 \subset K_2$ are two compact metric spaces, we have $\chi_{K_1} \subset \chi_{K_2}$.*

Proof. This easily follows from the equality

$$F(K_1^3) = (F(\pi_{12}^{-1})F(\pi_{12})F(K_1^3)) \cap (F(\pi_{13}^{-1})F(\pi_{13})F(K_1^3)),$$

which shows that each fiber of the map $\chi(F\pi^3 K_1)$ is also a fiber of the enveloping map $\chi(F\pi^3 K_2)$. □

Corollary 2.11.3. *$\chi_0(F) \subset \chi_K(F) \subset \chi(F)$ for every uncountable compact metric space K.*

Proposition 2.11.4. *Let F be a normal functor of degree n. Then $\chi_0(F) = \chi(F) = \chi_n(F)$.*

Proof. Suppose $K \in |\mathbf{Comp}|$ and $|K| \geq n$. Then $\chi_K(F) \supset \chi_n(F)$. If $a \in F(K^3)$, then $|\mathrm{supp}(a) \leq n|$ and we can find three sets A_1, A_2, A_3 with $|A_i| = n$, $A_i \supset \pi_i(\mathrm{supp}(a))$. Then $a \in F(A_1 \times A_2 \times A_3)$. The diagram

$$\begin{tikzcd} \mathcal{D} = F(A_1 \times A_2 \times A_3) \arrow[r] \arrow[d] & F(A_1 \times A_3) \arrow[d] \\ F(A_1 \times A_3) \arrow[r] & F(A_1) \end{tikzcd}$$

is isomorphic to the diagram $F(\pi^3(n))$, and since F preserves preimages and intersections, the set $(F(\pi^3(K)))^{-1}(a)$ coincides with the preimage of a under the characteristic map of the diagram $\mathcal{D}$. Thus, $\chi_K(F) \subset \chi_n(F)$.

□

Lemma 2.11.5. *For normal functors F', F'' we have $\chi_K(F' \times F'') = \{mn \mid m \in \chi_K(F'),\ n \in \chi_K(F'')\}$.*

By $\mathfrak{A}(m,n)$ we denote the set of $m \times n$-matrices $\alpha = (\alpha_{ij})$ whose elements belong to the set $\{0,1\}$. The set $\mathfrak{A}(m,n)$ is endowed with the partial order $\leq$:

$$\alpha \leq \beta \iff \alpha_{ij} \leq \beta_{ij},\ i = 1,\dots,m,\ j = 1,\dots,n.$$

For $\alpha \in \mathfrak{A}(m,n)$ denote by $Z(\alpha)$ the set of all $\beta \in \mathfrak{A}(m,n)$ satisfying the properties: $\alpha \leq \beta$, and every row and every column of β contains 1.

Proposition 2.11.6. *Let K be an uncountable metrizable compact space. Then*

$$\chi_K(\exp) = \{|Z(\alpha_1)| \dots |Z(\alpha_k)| \mid \alpha_1, \dots, \alpha_k \in \mathfrak{A}(m,n),\ k,m,n \in \mathbb{N}\} \cup \{\infty\}.$$

Proof. Let $\{x_1, \dots, x_k\} \in \exp_k X \setminus \exp_{k-1} X$, $\alpha_1, \dots, \alpha_k \in \mathfrak{A}(m,n)$. There exist closed sets $S_{ijp} \subset X$, $1 \le i \le m$, $1 \le j \le n$, $1 \le p \le k$, satisfying the conditions:

(1) x_p is a nonisolated point of S_{ijp};
(2) $S_{ijp} \cap S_{i'j'p} = \{x_p\}$, whenever $(i,j) \ne (i',j')$;
(3) $S_{ijp} \cap S_{i'j'p'} = \varnothing$, whenever $p \ne p'$.

Let $\alpha_p = (\alpha_p(i,j))$. Choose two sets $\{y_1, \dots, y_m\}$, $\{z_1, \dots, z_n\}$ in K of cardinality n and m respectively. Let

$$A = \bigcup\{S_{ijp} \times \{y_i\} \mid \alpha_p(i,j) = 1\} \cup \{\{x_p\} \times \{y_1, \dots, y_m\} \mid \alpha_p = 0\}$$
$$B = \bigcup\{S_{ijp} \times \{z_j\} \mid \alpha_p(i,j) = 1\} \cup \{\{x_p\} \times \{z_1, \dots, z_n\} \mid \alpha_p = 0\}.$$

Then $\exp \pi_1(A) = \exp \pi_1(B)$ and $|\chi_K^{-1}(A,B)| = |Z(\alpha_1)| \dots |Z(\alpha_k)|$. $\square$

The following example shows that, in general, $\chi_0(F) \ne \chi(F)$.

Example. We will use the following fact proved by M. Smurov [1985b]: $\chi_0(\exp\exp) \ni 37 \notin \chi_0(\exp)$.

Denote by $\mathcal{R}_X$ the equivalence relation on the space $(\exp X) \times X^{37}$ defined as follows:

$$(A, (x_1, \dots, x_{37}))\mathcal{R}_X(B, (y_1, \dots, y_{37}))$$

if and only if the set

$$A \cup \{x_1, \dots, x_{37}\} = B \cup \{y_1, \dots, y_{37}\}$$

is a continuum and there exists a permutation $\sigma \in S_{37}$ such that $y_i = x_{\sigma(i)}$.

Put $FX = ((\exp X) \times X^{37})/\mathcal{R}_X$ and let $qX\colon (\exp X) \times X^{37} \to FX$ be the quotient map. Given a map $f\colon X \to Y$ define the map $Ff\colon FX \to FY$ by the equality $Ff \circ qX = qY \circ ((\exp f) \times f^{37})$. Thus, we obtain a quotient functor F of the functor $\exp \times(-)^{37}$. It is easy to verify that F is a normal functor.

Since the natural transformation $(qX)_{X \in |\mathrm{Comp}|}$ is an isomorphism of the restrictions of the functors $\exp \times (-)^{37}$ and F onto the category $|\mathrm{Comp}_0|$ of zero-dimensional compacta, we have, by Lemma 2.11.5, $\chi_0(F) = \chi_0(\exp \times(-)^{37} = \chi_0(\exp)$. Consequently, $37 \notin \chi_0(F)$.

Consider the diagram $F(\pi^3 Q)$. Let $a \in Q$ and $I_1, I_2 \subset Q \times Q$ be a linear segments such that $\pi_1(I_1) = \pi_2(I_2) = \{a\}$. Suppose that $\{a_{ij} \mid j \in \omega\}$ is a countable dense subset in I_i, $i = 1, 2$. For every $j \in \omega$ and $i = 1, 2$ choose a linear segment $X_{ij} \subset Q \times Q$ so that the system $\{X_{ij} \mid i = 1, 2,\ j \in \omega\}$ satisfy the conditions:

1) a_{ij} is the endpoint of X_{ij};

2) the projection map π_1 embeds X_{ij} into Q and $\pi_1(X_{1j}) = \pi_1(X_{2j})$;

3) if $j \neq k$, then

$$\pi_1 X_{1j} \cap \pi_1 X_{1k} = \{a\} = \pi_1 X_{2j} \cap \pi_1 X_{2k};$$

4) $\lim_{j\to\infty} \operatorname{diam}(X_{ij}) = 0$.

Put $X_i = I_i \cup \bigcup\{X_{ij} \mid j \in \omega\}$, $Y = \pi_1(X_j)$. It is easy to see that there exists a unique closed subspace $Z \subset Q \times Q \times Q$ such that $\pi_{12}(Z) = X_1$, $\pi_{13}(Z) = X_2$. Obviously, Z is a continuum.

Let $b, c \in I_1$, $b \neq c$. Choose mutually distinct points $d_1, \dots, d_{37}$ and put

$$\begin{aligned}
u =&\mathcal{R}_Q(Y, (a, \dots, a)),\\
s =&\mathcal{R}_{Q\times Q}(X_1, (b, c, c, \dots, c)),\\
t =&\mathcal{R}_{Q\times Q}(X_2, (d_1, d_2, \dots, d_{37})).
\end{aligned}$$

It is easy to see that every point $r \in F(Q\times Q\times Q)$ such that $F\pi_{12}(r) = s$, $F\pi_{13}(r) = t$ is of the form

$$\mathcal{R}_{Q\times Q\times Q}(Z, (c, d_1), \dots, (c, d_{j-1}), (b, d_j), (c, d_{j+1}), \dots, (c, d_{37})),$$

for some j, $1 \leq j \leq 37$ (here by (x, y) we denote the point in $Q \times Q \times Q$ such that $\pi_{12}(x, y) = x$, $\pi_{13}(x, y) = y$). Thus, the preimage of the element $(s, t) \in F(Q \times Q) \times F(Q \times Q)$ consists of 37 points and $37 \in \chi(F)$.

Note that some other topological properties of the characteristic map can be considered. In particular, the *dimensional characteristic* $\dim \chi_K(F)$ of a functor F consists of the numbers $\dim((\chi)^{-1}(x))$.

Exercises

1. Prove counterparts of the properties of $\chi(F)$ for $\dim \chi(F)$.
2. (*E. Shchepin [1981]*) Prove that for every normal functor F of finite degree the foloowing holds: $\max \chi(SP^n F) = (\max \chi(F))^n \cdot n!$.

2.12. Notes and comments to Chapter 2

The most of results concerning hyperspaces and probability measures are classical. The construction of superextension is defined by J. de Groot [1967]; its functoriality is noticed by A. V. Ivanov, who also introduced the similar functors of full k-linked systems (A. Ivanov [1986]).The inclusion hyperspace functor is defined by E. V. Moiseev [1990]. The functors $\widetilde{G}$ and $\widetilde{N}_k$ and O are introduced by T. Radul [1990b] and [1998]. The projective power functors are defined by A. Szankowski [1970].

Definition 2.3.1 of normal functor is introduced in the fundamental paper E. Shchepin [1981]. Normality of the functors exp, P etc. is established in V. Fedorchuk [1981]. Propositions 2.2.1 and 2.2.2, and Theorem 2.2.3 are due to E. Shchepin [1981]. Proposition 2.2.7 is proved by V. Fedorchuk [1988].

Theorems 2.3.7 and 2.3.8 are due to M. Zarichnyi [1990a] Theorem 2.4.4 Proposition 2.4.7 is proved in M. Zarichnyi [1993].

Characterization Theorem 2.5.4 for the hyperspace and n-hyperspace functors is due to E. Shchepin [1981].

Propositions 2.6.1, 2.6.2 and 2.6.7 are proved by M. Zarichnyi [1990a]. For Theorem 2.6.12 see M. Zarichnyi [1992a].

Definition 2.6.15 and Theorem 2.6.16 are due to L. Shapiro [1988].

A. Chigogidze [1984] formulated a counterpart for the category **Tych** of the notion of normal functor, defined the extension F_β and proved Theorem 2.7.3. Proposition 2.7.9 and Theorem 2.7.11 are obtained by M. Zarichnyi [1990a].

The multiplicative functors are defined by E. Shchepin [1981]. Theorem 2.9.7, Corollary 2.9.8, and Theorem 2.9.10 are due to M. Zarichnyi [1987a].

The paper E. Shchepin [1981] contains the material of Section 2.10 with the following exeptions: Theorem 2.10.21 is proved by M. Zarichnyi [1992b], openness of the functors P, N_k and related functors by S. Ditor, R. Haydon [1976], A. Ivanov [1986].

Theorem 2.3.4 is proved by L. Shapiro [1992]. The notion of linear functorial operator extending pseudometrics is introduced by T. Banakh and O. Pikhurko [1997]; they proved Proposition 2.8.7, actually, in a more general form.

Strong bicommutativity and strong openness of N is proved by A. V. Ivanov [1986].

Chapter 3.

Normal monads

In this chapter we investigate monads in the category **Comp**, whose functorial parts are (weakly, almost) normal functors. Section 3.2 provides different examples of such monads. The problem of extension of (weakly, almost) normal functors onto the Kleisli categories of (weakly, almost) normal monads is considered in Section 3.5. The main concern of Section 3.6 is to give intrinsic characterizations of the categories of algebras; besides, in this section we deal with the problem of lifting of functors onto the categories of algebras. In Section 3.9 we continue to investigate connections between functors and metrics and consider the notions of uniformly metrizable functor and perfectly metrizable monad.

3.1. Normal monads

A monad $\mathbb{T} = (T, \eta, \mu)$ in the category **Comp** is called *(weakly, almost) normal* if so is the functor T. A similar definition can be also given for the category **Tych**.

Let $\mathbb{T} = (T, \eta, \mu)$ be a (weakly, almost) normal monad. For every countable ordinal α the transfinite iteration T^α of T and natural transformations

$$\mu_{\alpha,\beta}\colon T^\alpha \to T^\beta, \quad \alpha \geq \beta$$

are defined (see Section 1.2).

Proposition 3.1.1. *The functors T^α are (weakly, almost) normal.*

Suppose that $\mathcal{A}$ is a countable directed partially ordered set and for every $\alpha \in \mathcal{A}$ a (weakly, almost) normal monad $\mathbb{T}_\alpha = (T_\alpha, \eta_\alpha, \mu_\alpha)$ is given. Consider an inverse system $\mathcal{S} = \{T_\alpha, \varphi_{\alpha\beta}; \mathcal{A}\}$, where $\varphi_{\alpha\beta}$ is a morphism of the monad $\mathbb{T}_\alpha$ into the monad $\mathbb{T}_\beta$, $\beta \leq \alpha$.

Proposition 3.1.2. *In the category of (weakly, almost) normal monads in* **Comp** *there exists the limit of the inverse system* $\mathcal{S} = \{\mathbb{T}_\alpha, \varphi_{\alpha\beta}; \mathcal{A}\}$.

Definition 3.1.3. A monad $\mathbb{T} = (T, \eta, \mu)$ is called *open* if so is the natural transformation μ.

Exercise

1. Recall that for every (weakly, almost) normal functor F in **Comp** the subfunctor F^c of F is defined as follows:

$$F^cX = \{a \in FX \mid \operatorname{supp} a \text{ is contained in a connected component of } X\}$$

for every $X \in |\mathbf{Comp}|$. Suppose that F is the functorial part of a monad $\mathbb{T} = (T, \eta, \mu)$. Show that $\mu X(F^cF^cX) \subset F^cX$, i. e. F^c is the functorial part of a monad.

3.2. Some examples

3.2.1. Power monad

Let α be an ordinal, $1 \leq \alpha \leq \omega$. The unique natural transformation $\eta\colon \mathrm{Id} \to (-)^\alpha$ acts by the formula $\eta X(x) = (x, x, \dots) \in X^\alpha$ (the diagonal map). Define the natural transformation $\mu\colon (-)^\alpha(-)^\alpha \equiv (-)^{\alpha\times\alpha} \to (-)^\alpha$ by the formula $\mu X((x_{ij})_{i,j<\alpha}) = (x_{ii})_{i<\alpha}$. The proof of the following fact is left to the reader.

Proposition 3.2.1. *The triple* $\mathbb{T} = ((-)^\alpha, \eta, \mu)$ *is a normal monad in the category* **Comp**.

Note that the power monad is open.

Exercise

1. Show that there is the only monad whose functorial part is $(-)^\alpha$.

3.2.2. Hyperspace monad

The unique natural transformation from Id to exp will be denoted by s, $s(x) = \{x\}$ for $x \in X$ (the singleton map). Let $u\colon \exp^2 \to \exp$ be the natural transformation of union; the component $uX : \exp^2 X \to \exp X$ of u acts by the formula $uX(\mathcal{A}) = \bigcup \mathcal{A}$, $\mathcal{A} \in \exp^2 X$. It is easy to show that the triple $\mathbb{H} = (\exp, s, u)$ forms a monad. Note that $uX((\exp^c)^2 X) \subset \exp^c X$, for every X. This means that the triple $\mathbb{H}^c = (\exp^c, s, u^c = u|(\exp^c)^2)$ is a submonad of $\mathbb{H}$ (the *continuum hyperspace monad*).

Proposition 3.2.2. *The hyperspace monad is open.*

Proof. Indeed, this is a consequence of the equality

$$\begin{aligned} &uX(\langle\langle V_{11},\dots,V_{1n_1}\rangle,\dots,\langle V_{k1},\dots,V_{kn_k}\rangle\rangle) \\ &\quad = \langle V_{11},\dots,V_{1n_1},\dots,V_{k1},\dots,V_{kn_k}\rangle \end{aligned}$$

and the following fact: the sets of the form

$$\langle\langle V_{11},\dots,V_{1n_1}\rangle,\dots,\langle V_{k1},\dots,V_{kn_k}\rangle\rangle,$$

where V_{ij} are open in X, form a base of the topology in $\exp^2 X$. □

The following example shows that the continuum hyperspace monad is not open.

Examples. Consider the unit circle $S^1 \subset \mathbb{R}^2$. Let $\mathcal{A} = \{A \in \exp^c S^1 \mid \operatorname{diam}(A) = 1\}$ (we consider the metric on S^1 induced by the euclidean metric on $\mathbb{R}^2$. Then $u^c S^1(\mathcal{A}) = S^1$ and it is easy to see that $u^c S^1(\mathcal{U}) = \{S^1\}$ for sufficiently small neighborhood $\mathcal{U}$ of $\mathcal{A}$ in $(\exp^c)^2 S^1$. Thus, $u^c S^1$ is not open.

Theorem 3.2.3. *For the functor* exp *there exists a unique monad* $\mathbb{H} = (\exp, s, u)$ *in which it can be included.*

Proof. We have only to verify the uniqueness of the multiplication. Let ψ be the multiplication of some monad for exp.

We first show that for every $\mathcal{A} \in \exp^2 X$

$$\psi X(\mathcal{A}) \subset \bigcup\{a \in \exp X \mid a \in \mathcal{A}\}. \tag{3.1}$$

Let $B = \bigcup\{a \in \exp | a \in A\}$. We get from the definition of B that $A \subset \exp B$, and hence $A \in \exp^2 B$. Since ψ is a natural transformation, the diagram

$$\begin{tikzcd} \exp^2 B \arrow[r, "\psi B"] \arrow[d, hook] & \exp B \arrow[d, hook] \\ \exp^2 X \arrow[r, "\psi X"] & \exp X \end{tikzcd}$$

is commutative. Therefore, $\psi X(A) = \psi B(A) \in \exp B$, i.e., $\psi X(A) \subset B$, and inclusion (3.1) holds.

We verify the reverse inclusion

$$\psi X(A) \supset B \tag{3.2}$$

in several steps. For a single-element A, i.e., for an $A \in \exp X$, this is true, because ψX is a retraction (s is a left identity). Assume that we can prove (3.2) for finite $A \subset \exp X$. Then for arbitrary A inclusion (3.2) is obtained by passing to the limit in (3.2) for finite subsets $A' \subset A$, because they approximate A, and their unions $\bigcup A'$ approximate $\bigcup A$. It is assumed everywhere below that A is finite.

We now verify that the validity of (3.2) for zero-dimensional X implies its validity for arbitrary X. Let $f\colon X_0 \to X$ be an epimorphism of a zero-dimensional compact Hausdorff space X_0 onto X. Then the diagram

$$\begin{tikzcd} \exp^2 X_0 \arrow[r, "\psi X_0"] \arrow[d, "\exp^2 f"'] & \exp X_0 \arrow[d, "\exp f"] \\ \exp^2 X \arrow[r, "\psi X"] & \exp X \end{tikzcd} \tag{3.3}$$

is commutative. Let $A = \{a_1, \ldots, a_k\}$ be a finite subset of $\exp X$, and define $a_i^0 = f^{-1}a_i$ and $A_0 = \{a_1^0, \ldots, a_k^0\}$. Then, by assumption, $\psi X_0(A_0) = B_0 = a_1^0 \cup \cdots \cup a_k^0$, and $\exp f(\psi X_0(A_0)) = a_1 \cup \cdots \cup a_k$. The commutativity of diagram (3.3) implies that $a_1 \cup \cdots \cup a_k = \psi X(\exp^2 f(A_0))$. But $\exp^2 f(A_0) = A$. Therefore, $\psi X(A) = a_1 \cup \cdots \cup a_k$.

We now show that the validity of (3.2) for finite X implies its validity for zero-dimensional X. We represent a zero-dimensional compact Hausdorff space X as the limit of an inverse system $\mathcal{S} = \{X_\alpha, \pi_{\alpha\alpha'}; \alpha \in \mathcal{A}\}$ of finite spaces X_α. As before, suppose that $A = \{a_1, \ldots, a_k\} \subset \exp X$,

and let $B = a_1 \cup \cdots \cup a_k$. Assume that some point $x \in B$ does not belong to $\psi X(A)$. Then there exists a neighborhood Ox disjoint from $\psi X(A)$. There exist an $\alpha \in A$ and a neighborhood Ox_α of the point $x_\alpha = \pi_\alpha(x)$ such that $\pi_\alpha^{-1} Ox_\alpha \subset Ox$ (here π_α is the limit map of $\mathcal{S}$). Then the commutativity of the diagram

$$\begin{tikzcd}
\exp^2 X \arrow[r, "\psi X"] \arrow[d, "\exp^2 \pi_\alpha"'] & \exp X \arrow[d, "\exp \pi_\alpha"] \\
\exp^2 X_\alpha \arrow[r, "\psi X_\alpha"] & \exp X_\alpha
\end{tikzcd}$$

implies that

$$x_\alpha \notin \psi X_\alpha(\exp^2 \pi_\alpha(A)). \tag{3.4}$$

On the other hand, by our assumption, for $A_\alpha = \{\pi_\alpha(a_1), \ldots, \pi_\alpha(a_k)\}$ we have

$$\psi X_\alpha(A_\alpha) = \pi_\alpha(a_1) \cup \cdots \cup \pi_\alpha(a_k) = \pi_\alpha(B) \ni x_\alpha. \tag{3.5}$$

But $A_\alpha = \exp^2 \pi_\alpha(A)$, and hence, (3.4) contradicts (3.5).

It remains to verify inclusion (3.2) for finite X. Assume first that the sets a_i are disjoint and have the same number of points:

$$a_i = \{x_1^i, \ldots, x_n^i\}.$$

Assume that $x_j^i \notin \psi X(A)$ for some i and j, and consider any other point $x_{j_1}^{i_1}$. Then there exists a bijection $f\colon X \to X$ such that $f(x_j^i) = x_{j_1}^{i_1}$ and $\exp^2 f(A) = A$.

Indeed, if $i = i_1$, then it is necessary to take a bijection of a_i onto itself that carries x_j^i into $x_{j_1}^{i_1}$, and to supplement this by the identity map on $X \setminus a_i$ to form f. If $i \neq i_1$, then we can take a bijection g of the set $a_i \cup a_{i_1}$ onto itself such that $g(a_i) = a_i$, and $g(x_j^i) = x_{j_1}^{i_1}$. After this, as in the first case, it is necessary to supplement g by the identity map on $X \setminus a_i \cup a_{i_1}$.

The equality $\exp \circ \psi X = \psi X \circ \exp^2 f$ implies

$$\exp f \circ \psi X(A) = \psi X(A). \tag{3.6}$$

If we regard $C = \psi X(A)$ as a subset of X, then (3.6) is equivalent to the equality $C = f(C)$. On the other hand, $C = X \setminus \{x_j^i\}$. Then

$$f(C) \subset f(X \setminus \{x_j^i\}) = X \setminus \{x_{j_1}^{i_1}\}.$$

Thus, $x_{j_1}^{i_1} \notin f(C) = C$. Since the point $x_{j_1}^{i_1} \in B = a_1 \cup \cdots \cup a_k$ is arbitrary and since it has already been verified that $\psi X(A) \subset B$, the set $C = \psi X(A)$ is empty, a contradiction.

Now let $A = \{a_1, \ldots, a_k\}$ be arbitrary and suppose that $\psi X(A) \not\ni x$ for some $x \in B$. We embed X into the Hilbert cube Q. Then, as we noted at the end of the proof of (3.1), $\psi X(A) = \psi Q(A)$. We take disjoint neighborhoods O_1 and O_2 of the set $\psi Q(A)$ and a point x in Q, respectively. It follows from the continuity of ψQ that there is a neighborhood $OA \subset \exp^2 Q$ such that $\psi Q(A') \subset O_1$ for every $A' \subset OA$. In the neighborhood OA there exists an $A_0 = \{a_1^0, \ldots, a_k^0\}$ such that the a_i^0 are disjoint and have the same number of elements. Further, it can be assumed that $B_0 = a_1^0 \cup \cdots \cup a_k^0$ is contained in an arbitrary close neighborhood of B and $B_0 \cup O_2 \neq \varnothing$. Then $\psi B_0(A_0) = B_0$, by what was proved above. But $\psi B_0(A_0) = \psi Q(A_0)$. Accordingly, $B_0 = \psi Q(A_0) \subset O_1$, but this contradicts the fact that $B_0 \cap O_1 \neq \varnothing$ and $O_1 \cap O_2 = \varnothing$.

□

The monad $\mathbb{H}$ admits a functional description. Denote by $C_+(X)$ the space of non-negative continuous functions on X with natural metric, order, linear and multiplicative structure. For $\alpha \in \mathbb{R}$ let α_X be a constant function $\alpha(x) = \alpha$, $x \in X$.

Denote by $\Phi(X)$ the set of all functionals $\varphi\colon C_+(X) \to \mathbb{R}_+$ satisfying the following conditions:

1) $\varphi(f+g) \leq \varphi(f) + \varphi(g)$,
2) $\varphi(fg) \leq \varphi(f)\varphi(g)$,
3) $\varphi(f \leq g)$ implies $\varphi(f) \leq \varphi(g)$,
4) $\varphi(\alpha f) = \alpha\varphi(f)$,
5) $\varphi(f + \alpha_X) = \varphi(f) + \alpha$,
6) $\varphi(\alpha_X) = \alpha$,

where $f, g \in C_+(X)$, $\alpha \in \mathbb{R}_+$.

The set $\Phi(X)$ is endowed with the topology with a base consisting of sets of the form

$$O(\mu; \varphi_1, \ldots, \varphi_k; \varepsilon) = \{\mu' \in \Phi(X) \mid |\mu'(\varphi_i) - \mu(\varphi_i)| < \varepsilon \text{ for each } i = 1, \ldots, k\},$$

where $\varphi_1, \ldots, \varphi_k \in C_+(X)$, $\varepsilon > 0$. It is proved in L. Shapiro [1991] that the functor $\Phi(X)$ is naturally isomorphic to the hyperspace functor.

Consider the space $C(X, I)$ (recall that $I = [0,1]$) equipped with the sup-metric and the pointwise minimum semilattice operation.

Let $\mu \in C\big(C(X,I), I\big)$. The functional μ is called *normed* if $\mu(c_X) = c$ for every $c \in I$.

We say that μ is *supported* on a closed set $A \subset X$ if for every functions $g_1, g_2 \colon X \to I$ with $g_1|A = g_2|A$ we have $\mu(g_1) = \mu(g_2)$.

A minimal closed set A on which μ is supported is said to be a *support of* μ (briefly, $A = \operatorname{supp}(\mu)$). So, we can consider μ as an element of $C\big(C(A, I), I\big)$ whenever $A = \operatorname{supp}(\mu)$.

The functional μ is called *symmetric on its support* if for every $\varphi \in C(A, I)$, $h \in \operatorname{Auth}\big(\varphi(A)\big)$ we have $\mu(\varphi) = \mu(h \circ \varphi)$. Finally we say that μ *preserves the semilattice operation* if

$$\mu(\min\{\varphi_1, \varphi_2\}) = \min\{\mu(\varphi_1), \mu(\varphi_2)\}$$

for all $\varphi_1, \varphi_2 \in C(X, I)$.

Consider the space EX consisting of $\mu \in C\big(C(X,I), I\big)$ which are normed, symmetric on its support, and preserve the semilattice operation. The space EX provides a topology whose base is formed by the sets $O(\mu; \varphi_1, \dots, \varphi_k; \varepsilon)$, where $\varphi_1, \dots, \varphi_k \in C(X, I)$.

Let $f \colon X \to Y$ be a map and a map $f^* \colon C(Y, I) \to C(X, I)$ send $\varphi \in C(Y, I)$ to $f^*(\varphi)$, $f^*(\varphi) = \varphi\big(f(x)\big)$, $x \in X$. Define a map $Ef \colon EX \to EY$ by the formula

$$Ef(\mu)(\varphi) = \mu\big(f^*(\varphi)\big),$$

where $\mu \in EX$, $\varphi \in C(Y, I)$. Below we shall prove that EX is a compact Hausdorff space. Hence, E is an endofunctor on the category **Comp**.

Define maps $\eta X \colon X \to EX$ and $\mu X \colon E^2X \to EX$ by the following formulae:

$$\begin{aligned} \eta X(x)(\varphi) &= \varphi(x), \quad & x \in X, \\ \mu X(\alpha)(g) &= \alpha(\tilde{g}), \quad & \alpha \in E^2X,\ g \in C(X, I), \end{aligned}$$

where $\tilde{g} \colon EX \to I$ is the map $\tilde{g}(\mu) = \mu(g)$, $\mu \in EX$.

It is easy to verify that the families $\{\eta X\}$ and $\{\mu X\}$ form natural transformations η and μ, respectively, and moreover, the triple $\mathbb{E} = (E, \eta, \mu)$ is a monad on **Comp**.

We shall prove that the monad $\mathbb{E} = (E, \eta, \mu)$ is isomorphic to the hyperspace monad.

Consider a map $tX \colon \exp X \to EX$, $t(A)(f) = \inf f(A)$, $f \in C(X, I)$.

Lemma 3.2.4. *The map tX is a homeomorphism of* $\exp X$ *onto EX.*

Proof. Let $tX(A) = \mu \in EX$ and $O(\mu; \varphi_1, \dots, \varphi_n; \varepsilon)$ be a neighborhood of μ. Choose an open cover $O_1, \dots, O_k$ of A such that $\operatorname{diam} \varphi_i(O_j) < \varepsilon$ for all $i = 1, \dots, n$, $j = 1, \dots, k$. Then

$$tX(\langle O_1, \dots, O_k \rangle) \subset O(\mu; \varphi_1, \dots, \varphi_n; \varepsilon)$$

and, hence, the map tX is continuous.

Consider distinct $A_1, A_2 \in \exp X$. We may suppose that there exists a point $a \in A_1 \setminus A_2$. Let $f \in C(X, I)$ be a function such that $f(a) = 0$ and $f(A_2) = 1$. Then $tX(A_1)(f) = 0 < 1 = tX(A_2)(f)$. Hence, tX is injective.

Prove the surjectivity of tX. Let $\nu \in EX$. We may assume that $\operatorname{supp}(\nu) = X$. So, we have to show that $\nu(f) = \inf f(X)$. Without restriction of generality, one can suppose that $\inf f(X) = 0$.

To the contrary, let $\nu(f) = a > 0$. Since ν is normed, there exists a point $x \in X$ with $f(x) \geq a$. Considering the function $\inf\{f, a_X\}$ at the necessarity, one can suppose that $f(x) \leq a$ for all $x \in X$.

If $f(X) = [0, a]$, define a homeomorphism $h\colon [0, a] \to [0, a]$ by the formula $h(t) = a - t$, $t \in [0, a]$. The symmetry condition implies $\nu(h \circ f) = \nu(f) = a$. Put $g = \inf\{h \circ f, f\}$. Then we have $\nu(g) = \inf\{\nu(h \circ f), \nu(f)\} = a$ but $g(x) \leq \frac{1}{2}a$ for every $x \in X$. Hence, $\nu(g) \leq \frac{1}{2}a$ and we obtain a contradiction.

Now let $f(X) \neq [0, a]$. There exists a point $b \in (0, a)$ such that $b \notin f(X)$. Consider a function $f_1 : X \to \{b, a\}$ defined by the formula:

$$f_1(x) = \begin{cases} a, & f(x) > b, \\ b, & f(x) < b. \end{cases}$$

Since $f \leq f_1 \leq a_X$, we have $\nu(f_1) = a$. Let

$$f_2(x) = \begin{cases} b, & f(x) > b, \\ a, & f(x) < b \end{cases}$$

be a function $X \to \{a, b\}$.

By the symmetry condition, $\nu(f_2) = \nu(f_1) = a$. But $\inf\{f_1, f_2\} = b_X$ and consequently $\nu(\inf\{f_1, f_2\}) = b$. We obtain a contradiction again.

Thus the map tX is a homeomorphism. □

By this lemma, EX is a compact Hausdorff space.

Theorem 3.2.5. *The natural transformation $t = \{tX\}$ is an isomorphism of the hyperspace monad $\mathbb{H}$ into the monad $\mathbb{E} = (E, \eta, \mu)$.*

Proof. Show that t is a natural transformation. Let $f\colon X \to Y$ be a map and $A \in \exp X$. Then we have

$$tY \circ \exp f(A) = \inf \varphi(f(A)), \quad \varphi \in C(Y, I)$$

and

$$Ef \circ tX(A)(\varphi) = \inf f^*(\varphi) = \inf \varphi(f(A)).$$

Now prove that t is a morphism of $\mathbb{H}$ to $\mathbb{E}$. The equality $t \circ s = \eta$ is obvious. Verify that $t \circ u = \mu \circ tE \circ \exp t$. Take any $\mathcal{A} \in \exp^2 X$ and $\varphi \in C(X, I)$. Then $t \circ u(\mathcal{A})(\varphi) = \inf \varphi(\bigcup \mathcal{A})$ and

$$\mu \circ tE \circ \exp t(\mathcal{A})(\varphi) = tE \circ \exp t(\mathcal{A})(\widetilde{\varphi}) = \inf\{\inf \varphi(A) \mid A \in \mathcal{A}\} = \inf \varphi(\bigcup \mathcal{A}).$$

□

Now the statement of the theorem follows from the previous lemma.

Exercises

1. Show that there exists a unique monad in **Comp**whose functorial part is the continuum hyperspace functor.
2. Are there the identity monad and the continuum hyperspace monad the only submonads of $\mathbb{H}$?

3.2.3. Inclusion hyperspace monad and their submonads

Define the map $\mu X = \mu_G X : G^2 X \to GX$ by the formula

$$\mu_G X(\mathfrak{A}) = \bigcup\{\bigcap \mathcal{M} \mid \mathcal{M} \in \mathfrak{A}\}, \ \mathfrak{A} \in G^2 X.$$

It follows from Propositions 2.1.3 and 2.1.8 that the map $\mu_G X$ is continuous.

Lemma 3.2.6. *Suppose that $C \in \exp X$. Then $\mu X^{-1}(C^+) = C^{++}$ and $\mu X^{-1}(C-) = C^{--}$.*

Lemmas 2.1.14 and 3.2.6 yield

Lemma 3.2.7. *If $A \in GX$, then*

$$\mu X^{-1}(\mathcal{A}) = \left(\bigcap\{A^{++} \mid A \in \mathcal{A}\}\right) \cap \left(\bigcap\{B^{--} \mid B \in \perp X(\mathcal{A})\}\right).$$

Proof. If $\mathfrak{A} \in \mu X^{-1}(C^+)$, then there exists an $\mathcal{M} \in GA$ such that $C \in \bigcap \mathcal{M}$. The latter means that $\mathcal{A} \in C^{++}$ for all $\mathcal{A} \in \mathcal{M}$, i.e., $\mathcal{M} \in C^+$ or $\mathfrak{A} \in C^{++}$.

If $\mathfrak{A} \in \mu X^{-1}(C^-)$, then $\bigcap \mathcal{M} \in C^-$ for all $\mathcal{M} \in \mathfrak{A}$. But by lemma 2.1.10, the latter means that $\mathcal{M} \cap C^- \neq \varnothing$, i.e., $\mathfrak{A} \in C^-$. □

Lemma 3.2.8. *$\mu = (\mu X)_{X \in \mathbf{Comp}}$ is a natural transformation of G^2 to G.*

Proof. Let $\mathcal{A} \in GX$. For a map $f : X \to Y$ and $\mathfrak{A} \in G^2 X$ let

$$\begin{aligned}
\mathcal{B}_1 =& \mu Y \circ G^2 f(\mathfrak{A}) = \mu Y \circ rGX \circ \exp^2 Gf(\mathfrak{A}) \\
=& \mu Y \circ rGY(\{\{Gf(\mathcal{A}) \mid \mathcal{A} \in \alpha\} \mid \alpha \in \mathfrak{A}\} \\
=& \mu Y(\{\beta \in \exp GY \mid \beta \supset \{Gf(\mathcal{A}) \mid \mathcal{A} \in \alpha\}, \ \alpha \in \mathfrak{A}\}, \\
\mathcal{B}_1 =& Gf \circ \mu X(\mathfrak{A}) = rGY \circ \exp^2 f(\cup\{\cap\alpha \mid \alpha \in \mathfrak{A}\}) \\
=& \{B \in \exp Y \mid B \supset f(A), \ A \in \cup\{\cap\alpha \mid \alpha \in \mathfrak{A}\}\}.
\end{aligned}$$

If $B \in \mathcal{B}_1$ then for some $\alpha \in \mathfrak{A}$ we have $B \in \cap Gf(\alpha)$, and hence $B \in Gf(\mathcal{A})$ for all $\mathcal{A} \in \mathcal{A}$. This implies that for every $\mathcal{A} \in \alpha$ there exists $A \in \mathcal{A}$, for which $B \supset f(A)$, i. e. $B \in \mathcal{B}_2$. Hence $\mathcal{B}_1 \subset \mathcal{B}_2$.

To prove the inverse inclusion, consider $B \in \mathcal{B}_2$. Then there exists $\alpha \in \mathfrak{A}$ with the property that for every $\mathcal{A} \in \alpha$ there exists $A \in \mathcal{A}$ such that $B \supset f(A)$. Thus, $B \in Gf(\mathcal{A})$ for every $\mathcal{A} \in \alpha$ and, for $\beta = Gf(\alpha)$, we obtain $B \in \cap\beta$, and therefore $B \in \mathcal{B}_1$, i. e. $\mathcal{B}_2 \subset \mathcal{B}_1$. □

Note that the map $\eta X : X \to GX$ acts by the formula

$$\eta X(x) = \{\{x\}\}, \quad x \in X.$$

Proposition 3.2.9. *The triple* $\mathbb{G} = (G, \eta, \mu)$ *is a weakly normal monad in* **Comp**.

Proof. Let $\mathcal{A} \in GX$, then

$$\begin{aligned}
\mu X \circ \eta GX(\mathcal{A}) =& \mu X(\{\beta \in \exp GX \mid \mathcal{A} \in \beta\}) \\
=& \cup \{\cap\beta \mid \beta \in \exp GX, \ \mathcal{A} \in \beta\} = \mathcal{A}, \\
\mu X \circ G\eta X(\mathcal{A}) =& \mu X \circ rX \circ \exp^2 \eta X(\mathcal{A}) \\
=& \cup \{\cap\beta \mid \beta \in \exp GX, \ \beta \supset \{\eta X(a) \mid a \in A\}, \ A \in \mathcal{A}\} \\
=& \mathcal{A},
\end{aligned}$$

i. e. $\mu \circ \eta G = \mu \circ G\eta = 1_G$.

To prove the equality $\mu \circ \mu G = \mu \circ G\mu$, consider $\mathfrak{A} \in G^3X$ and let $\mathcal{A}_1 = \mu \circ \mu G(\mathfrak{A})$, $\mathcal{A}_2 = \mu \circ G\mu(\mathfrak{A})$. It is easy to see that

$$\begin{aligned}
\mathcal{A}_1 =& \cup \{\cap\beta \mid \beta \in \exp GX, \\
& \beta \supset \{\cup\{\cap\alpha \mid \alpha \in \mathfrak{A}\} \mid \mathfrak{A} \in \tilde{\alpha} \mid \tilde{\alpha} \in \tilde{\mathfrak{A}}\}, \\
\mathcal{A}_2 =& \cup \{\cap\alpha \mid \alpha \in \cup\{\cap\tilde{\alpha} \mid \tilde{\alpha} \in \tilde{\mathfrak{A}}\}\}.
\end{aligned}$$

Let $A \in \mathcal{A}_2$, then there exists $\tilde{\alpha} \in \exp G^2$ such that for every $\mathfrak{A} \in \tilde{\alpha}$ there exists $\alpha \in \mathfrak{A}$ with the property that $A \in \mathcal{A}$ for every $\mathcal{A} \in \alpha$. Thus, $A \in \mathcal{A}_1$ and $\mathcal{A}_1 \subset \mathcal{A}_2$.

The opposite inclusion can be directly verified. □

Note that the functor G can be included in a monad $(G, \eta, \mu') \neq (G, \eta, \mu)$. Indeed, one can take $\mu' = \perp \circ \mu \circ G \perp \circ \perp G$.

Lemma 3.2.10. *For every $X \in |\mathbf{Comp}|$ we have $\mu X(N_k^2 X) \subset N_k X$ and $\mu X(\lambda^2 X) \subset \lambda X$.*

Proof. Let $\mathfrak{A} \in N_k^2 X$ and $A_1, \dots, A_k \in \mu X(\mathfrak{A})$. Then there exist $\alpha_i \in \mathfrak{A}$ such that $A_i \in \cap\alpha_i$, $1 \le i \le k$. Choose $\mathcal{A} \in \alpha_1 \cap \dots \cap \alpha_k \neq \varnothing$, then $A_i \in \mathcal{A}$ for every i, $1 \le i \le k$. Since $\mathcal{A} \in N_k X$, we obtain $A_1 \cap \dots \cap A_k \neq \varnothing$. This means that $\mu X(\mathfrak{A}) \in N_k X$.

Now suppose that $\mathfrak{A} \in \lambda^2 X$. We have just proved that $\mu X(\mathfrak{A}) \in NX$.

First show that for every $A \in \exp X$ with $\mu X(\mathfrak{A}) \in A^+$ we have $\mathfrak{A} \in A^{++}$. Indeed, otherwise $\alpha \cap A_i^+ = \varnothing$ for some $\alpha \in \mathfrak{A}$, whence $\alpha \subset (X \setminus A)^+$ and then it is easy to deduce that $\alpha \subset B^+$ for some $B \in \exp X$, $B \cap A = \varnothing$. This implies that $B \in \mu X(\mathfrak{A})$ and we obtain a contradiction.

If $\mu X(\mathfrak{A}) \notin \lambda X$, there exist $A_1, A_2 \in \exp X$, $A_1 \cap A_2 = \varnothing$, for which $\mu X(\mathfrak{A}) \cup \{A_i\} \in NX$, $i = 1, 2$. But then $\mathfrak{A} \in A_i^{++}$, $i = 1, 2$, whence $A_1^+ \cap A_2^+ = \varnothing$ and consequently $A_1 \cap A_2 = \varnothing$. □

Since $\eta X(X) \subset \lambda X$, , the triples $\mathbb{N}_k = (N_k, \eta, \mu|N_k^2)$ and $\mathbb{L} = (\lambda, \eta, \mu|\lambda^2)$ are weakly normal submonads of the monad $\mathbb{G}$.

3.2.4. Monads generated by the functors $\widetilde{G}, \widetilde{N}_k$

There are normal counterparts of the monads $\mathbb{G}$ and $\mathbb{N}_k$.

For a compact Hausdorff space X let $\widetilde{\cup}, \widetilde{\cap}\colon \exp \widetilde{G}X \to \widetilde{G}X$ be the following maps:

$$\widetilde{\cup}(\mathfrak{A}) = \gamma X(\bigcup \mathfrak{A}),$$
$$\widetilde{\cap}(\mathfrak{A}) = \bigcap\{\gamma X(\mathcal{A} \cup \{\bigcup\{\bigcup \mathcal{B} \mid \mathcal{B} \in \mathfrak{A}\}\}) \mid \mathcal{A} \in \mathfrak{A}\}, \quad \mathfrak{A} \in \exp \widetilde{G}X,$$

where the retraction $\gamma X\colon \exp^2 X \to \widetilde{G}X$ is defined in subsection 2.1.4. We leave to the reader verifying continuity of these maps.

Define the natural transformations $\eta\colon \mathrm{Id} \to \widetilde{G}$ and $\mu\colon \widetilde{G}^2 \to \widetilde{G}$ by the formulae:

$$\eta X(x) = \{\{x\}\},$$
$$\mu X(\mathfrak{A}) = \widetilde{\cup}(\widetilde{\cap}\mathcal{A} \mid \mathcal{A} \in \mathfrak{A}), \ \mathfrak{A} \in \widetilde{G}^2 X, \ X \in \mathbf{Comp}.$$

Proposition 3.2.11. *The triples $(\widetilde{G}, \eta, \mu)$ and $(\widetilde{N}_k, \eta, \mu|\widetilde{N}_k^2)$ form normal monads in* **Comp**.

Proof. The associativity and two-side unity can be checked by straightforward computations. □

3.2.5. Probability measure monad

Let $X \in |\mathbf{Comp}|$. Denote by $uX\colon C(X) \to C(PX)$ the map defined by the formula $uX(\varphi)(\mu) = \mu(\varphi)$. The map uX is a linear operator with $\|uX\| = 1$. Define the map $\psi X\colon P^2X \to PX$ by the formula $\psi X(M)(\varphi) = M(uX(\varphi))$, $\varphi \in C(X)$.

Recall that $\eta X(x) = \delta_x$ (the Dirac measure concentrated in $x \in X$).

Proposition 3.2.12. *The triple* $\mathbb{P} = (P, \eta, \psi)$ *is a monad in* **Comp**.

Proof. We have ($\varphi \in C(X)$, $\mu \in PX$)

$$\psi X \circ \eta PX(\mu)(\varphi) = \eta PX(\mu)(uX(\varphi)) = uX(\varphi)(\mu) = \mu(\varphi),$$

i. e. $\psi \circ \eta P = 1_P$;

$$\psi X \circ P\eta X(\mu)(\varphi) = P\eta X(\mu)(uX(\varphi)) = \mu(uX(\varphi) \circ \eta X) = \mu(\varphi),$$

i. e. $\psi \circ P\eta = 1_P$.

For every $M \in P^2X$ and $\varphi \in C(X)$ we have

$$uPX \circ uX(\varphi)(M) = M(uX(\varphi)) = psiX(M)(\varphi) = uX(\varphi) \circ \psi X(M).$$

This implies that for every $\mathfrak{M} \in P^3X$ and $\varphi \in C(X)$ we have

$$\begin{aligned}\psi X \circ \psi PX(\mathfrak{M})(\varphi) =& \psi PX(\mathfrak{M})(uX(\varphi)) = \mathfrak{M}(uPX \circ uX(\varphi)) = \\ \mathfrak{M}(uX(\varphi) \circ \psi X) =& P\psi X(\mathfrak{M})(uX(\varphi)) = \psi X \circ P\psi X(\mathfrak{M})(\varphi),\end{aligned}$$

i. e. $\psi \circ \psi P = \psi \circ P\psi$. □

Proposition 3.2.13. *The monad* $\mathbb{P}$ *is the only monad whose functorial part is* P.

Proof. Let $m, n \in \omega$ and

$$M = \frac{1}{m}\sum_{i \in m} \delta_{\mu_i} \in P^2(m \times n), \quad \mu_i = \frac{1}{n}\sum_{j \in n} \delta_{(i,j)}. \tag{3.7}$$

Suppose that $h\colon m \times n \to m \times n$ is a homeomorphism such that $\mathrm{pr}_1 \circ h = g \circ \mathrm{pr}_1$ for some homeomorphism $g\colon m \to m$ (i. e. h is fibrewise with respect to pr_1). Then, obviously, $P^2h(M) = M$.

If $\mathbb{P}' = (P, \eta, \psi')$ is a monad, then, by naturality of ψ', we have $Ph \circ \psi'(m \times n)(M) = \psi'(m \times n)(M)$. Thus, the element $\psi'(m \times n)(M)$ is invariant with respect to all homeomorphisms Ph, where h is as above. From this, it is easy to deduce that

$$\psi'(m \times n)(M) = \frac{1}{m \times n} \sum_{i \in m,\ j \in n} \delta_{(i,j)} = \psi(m \times n)(M).$$

Now, suppose that $M' \in P^2_\omega X$,

$$M' = \sum_{i=1}^{k} \alpha_i \delta_{\mu'_i}, \quad \mu'_i = \sum_{j=1}^{l_i} \beta_{ij} \delta_{x_{ij}},$$

where α_i, β_j are rational. Then there exist $m, n \in \omega$, a probability measure $M \in P^2(m \times n)$ of form (3.7), and a map $f\colon m \times n \to X$ such that $M' = P^2 f(M)$. Then we have

$$\begin{aligned}\psi' X(M') =& \psi' X \circ P^2 f(M) = Pf \circ \psi'(m \times n)(M) = \sum_{i=1}^{k} \sum_{j=1}^{l_i} \alpha_i \beta_{ij} \delta_{x_{ij}} \\ =& \psi(M').\end{aligned}$$

Since such M' are dense in P^2X, we see that $\psi X = \psi' X$. □

Define the map $\mu X\colon O^2X \to OX$ by the formula $\mu X(\alpha)(g) = \alpha(\tilde{g})$, where $\alpha \in O^2X$, $g \in C(X, [0,1])$ and the map $\tilde{g}\colon OX \to [0,1]$ is defined by the formula $\tilde{g}(\mu) = \mu(g)$, $\mu \in OX$. It is easy to check that μX is well-defined and continuous.

Put $\eta X = \delta$. It is easy to check that ηX and μX are the components of the natural transformations $\eta\colon \mathrm{Id} \to O$ and $\mu\colon O^2 \to O$.

Theorem 3.2.14. *The triple $\mathbb{O} = (O, \eta, \mu)$ forms a monad on the category* **Comp**.

Proof. Let $\nu \in OX$. Consider any $\varphi \in C(X)$. Then we have

$$\mu X \circ \eta OX(\nu)(\varphi) = \eta OX(\nu)(\tilde{\varphi}) = \tilde{\varphi}(\nu) = \nu(\varphi)$$

and

$$\mu X \circ O\eta X(\nu)(\varphi) = O\eta X(\nu)(\tilde{\varphi}) = \nu(\tilde{\varphi} \circ \eta X) = \nu(\varphi).$$

Now let $\mathcal{N} \in O^3 X$ and $\varphi \in C(X)$. Then

$$\mu X \circ \mu O X(\mathcal{N})(\varphi) = \mu O X(\mathcal{N})(\tilde{\varphi}) = \mathcal{N}(\tilde{\tilde{\varphi}})$$

and

$$\mu X \circ O\mu X(\mathcal{N})(\varphi) = O\mu X(\mathcal{N})(\tilde{\varphi}) = \mathcal{N}(\tilde{\varphi} \circ \mu X) = \mathcal{N}(\tilde{\tilde{\varphi}}),$$

where $\tilde{\tilde{\varphi}} \in C(O^2 X)$ is defined by the formula $(\tilde{\tilde{\varphi}})(\nu) = \nu(\tilde{\varphi})$, $\nu \in O^2 X$. □

Remark 3.2.15. It is easy to check that the monad $\mathbb{P}$ is a submonad of $\mathbb{O}$.

Exercises

1. Consider the functor $\mathrm{cc}\, P$. The elements of the form $\mathrm{conv}\{\mu_1, \dots, \mu_k\}$ form a dense subset in $\mathrm{cc}\, PX$, for every $X \in |\mathbf{Comp}|$. Let $\mathcal{K} \in (\mathrm{cc}\, P)^2 X$, $\mathcal{K} = \mathrm{conv}\{A_i \mid i = 1, \dots, n\}$, where $A_i \in P\,\mathrm{cc}\, PX$, $A_i = \sum_{j=1}^{s_i} \alpha_{ij}\delta_{L_{ij}}$, $L_{ij} = \mathrm{conv}\{\mu_{ijk} \mid k = 1, \dots, t_{ij}\}$. Let

$$\psi X(\mathcal{K}) = \mathrm{conv}\{\textstyle\sum\{\alpha_{ij} m_{ijq_i(j)} \mid j = 1, \dots, s_i,$$
$$q_i : \{1, \dots, s_i\} \to \{1, \dots, t_{ij}\} \text{ is a function }\} \mid i = 1, \dots, n\}.$$

Show that this map ψX uniquely determines a continuous map from $(\mathrm{cc}\, P)^2 X$ to $\mathrm{cc}\, PX$ (which we also denote by ψX).

Show that $\psi = (\psi X)$ is a natural transformation from $(\mathrm{cc}\, P)^2$ to $\mathrm{cc}\, PX$ and the triple $(\mathrm{cc}\, P, \eta, \psi)$ is a monad in $\mathbf{Comp}$. Show that the probability measure monad can be considered as a submonad of $(\mathrm{cc}\, P, \eta, \psi)$.

2. Is the functor $G_{\mathrm{cc}} P$ the functorial part of a monad in $\mathbf{Comp}$?

3.3. Continuum of normal monads

In this section we will construct a continuum mutually nonisomorphic normal monads.

Recall, that the triple $\mathbb{H} = (\exp, s, u)$, $sX(x) = \{x\}$, $uX(\mathcal{A}) = \bigcup \mathcal{A}$, $x \in X$, $\mathcal{A} \in \exp^2 X$, $X \in |\mathbf{Comp}|$, forms a normal monad (the hyperspace monad). Consider a triple $\mathbb{H}_\omega = (T, \eta, \mu)$,

$$T = (\exp(-))^\omega, \quad \eta = (s)_{i=1}^\infty, \quad \mu = \prod_{i=1}^\infty (u \circ \exp \pi_i),$$

where $\pi_i = (\pi_i X)\colon (\exp(-))^\omega \to \exp$ is the natural projection on the i-th factor.

One can easily prove the following result.

Lemma 3.3.1. *The triple $\mathbb{H}_\omega$ forms a normal monad.*

Consider some order $\preccurlyeq$ on the set $\mathbb{N}$ of natural numbers and define a subfunctor $F_\preccurlyeq$ of $(\exp(-))^\omega$ in the following way:

$$F_\preccurlyeq X = \{(A_i)_{i=1}^\infty \in (\exp X)^\omega \mid i \preccurlyeq j \implies A_i \subset A_j\}, \qquad X \in |\mathbf{Comp}|.$$

It is easy to prove that $F_\preccurlyeq$ is a normal subfunctor of $(\exp(-))^\omega$ (for this it is sufficient to verify that $F_\preccurlyeq$ is epimorphic; see Section 2.3).

Proposition 3.3.2. *For every order $\preccurlyeq$ on $\mathbb{N}$ the triple*

$$\mathbb{F}_\preccurlyeq = (F_\preccurlyeq, \eta, \mu | F_\preccurlyeq^2)$$

forms a normal monad.

Proof. It is sufficient to prove that $\eta X(X) \subset F_\preccurlyeq X$ and $\mu X(F_\preccurlyeq^2 X) \subset F_\preccurlyeq X$, $X \in |\mathbf{Comp}|$. The first inclusion is evident. To obtain the second one consider an arbitrary point $(\mathcal{A}_i)_{i=1}^\infty \in F_\preccurlyeq^2 X$. Every point $a \in \mathcal{A}_i$ can be written as $\left(A_n^a(i)\right)_{n=1}^\infty$, $A_n^a(i) \in \exp X$. Denote by U_i the set $\bigcup\{A_i^a(i) \mid a \in \mathcal{A}_i\}$. Then

$$\mu X\left((\mathcal{A}_i)_{i=1}^\infty\right) = \left((uX \circ \exp \pi_i)(\mathcal{A}_i)\right)_{i=1}^\infty = (U_i)_{i=1}^\infty.$$

Let $i \preccurlyeq j$. It is sufficient to show that $U_i \subset U_j$. Let $x \in U_i$ be arbitrary. Then $x \in A_i^{a_0}(i)$ for some $a_0 \in \mathcal{A}_i$. Since $\mathcal{A}_i \subset \mathcal{A}_j$, we have $A_k^{a_0}(i) = A_k^{a'}(j)$, $k = 1, \dots, \infty$, for some $a' \in \mathcal{A}_j$. Therefore

$$x \in A_i^{a_0}(i) \subset A_j^{a_0}(i) = A_j^{a'}(j) \subset U_j.$$

$\square$

Remark 3.3.3. If orders $\preccurlyeq_1$ and $\preccurlyeq_2$ on $\mathbb{N}$ are isomorphic, then monads $\mathbb{F}_{\preccurlyeq_1}, \mathbb{F}_{\preccurlyeq_2}$ are also isomorphic. Indeed, for a given isomorphism $d\colon (\mathbb{N}, \preccurlyeq_1) \to (\mathbb{N}, \preccurlyeq_2)$ an isomorpism φ of $\mathbb{F}_{\preccurlyeq_1}$ and $\mathbb{F}_{\preccurlyeq_2}$ can be formed in such a way:

$$\varphi X\left((A_i)_{i=1}^\infty\right) = (A_{d^{-1}(i)})_{i=1}^\infty, \quad (A_i)_{i=1}^\infty \in F_{\preccurlyeq_1} X, \quad X \in |\mathbf{Comp}|.$$

Theorem 3.3.4. *Let $\preccurlyeq_1$ and $\preccurlyeq_2$ be orders on $\mathbb{N}$. Then any isomorphism of functors $F_1 = F_{\preccurlyeq_1}$ and $F_2 = F_{\preccurlyeq_2}$ implies existence of an isomorphism of the orders $\preccurlyeq_1$ and $\preccurlyeq_2$.*

We first prove the following lemma.

Lemma 3.3.5. *Under the conditions of Theorem 3.3.4, suppose that $\varphi = (\varphi X)\colon F_1 \to F_2$ be an isomorphism. Let X be the Alexandrov compactification of $\mathbb{N}$ with the limit point 0 and $(\Delta_i)_{i=1}^{\infty}$ a disjoint partition of $\mathbb{N}$ by infinite sets. If*

$$K_i = \bigcup\{\Delta_j \mid j \preccurlyeq_1 i\} \cup \{0\}, \ K = (K_i)_{i=1}^{\infty} \ \text{and} \ L = (L_i)_{i=1}^{\infty} = \varphi X(K);$$

then every set K_i, $i \in \mathbb{N}$, is contained among the sets L_i, $i \in \mathbb{N}$.

Proof. Fix any $n \in \mathbb{N}$. Show that the set K_n belongs to the family of sets L_i, $i \in \mathbb{N}$.

Remark first, that if $L_i \cap \Delta_j \neq \varnothing$, then $\Delta_j \subset L_i$. Indeed, the point K is invariant with respect to the identical on $X \setminus \Delta_j$ authomorphisms of X, therefore the point L also possesses this property.

If some set L_m contains Δ_i, then $L_m \supset K_i$. Indeed, let $j \prec_1 i$. Considering some points $x \in \Delta_i$, $y \in \Delta_j$, we construct a map $p\colon X \to X$ by the following formulae $p(x) = y$, $p(t) = t$ for other $t \in X$. Then the point $F_1 p(K)$ is invariant with respect to the identical on $X \setminus \Delta_j$ authomorphisms of X. Hence, the point $F_2 p(L)$ is such one. Therefore, $L_m \cap \Delta_j \neq \varnothing$ and $L_m \supset \Delta_j$. Moreover, since the set L_m is an infinite compact Hausdorff space, it contains 0.

Suppose that the family of sets L_m, $m \in \mathbb{N}$, does not contain K_n.

Remark that the set of numbers m with $L_m \supset \Delta_n$ is nonempty. Otherwise, one can consider the map $p\colon X \to X$, $p(x) = y$, $p(t) = t$ for $t \notin \{x, y\}$, where x, y are fixed points of Δ_n, $x \neq y$. Then the point $F_2 p(L)$ is invariant with respect to the identical on $X \setminus \Delta_n$ authomorphisms of X, but the point $F_1 p(K)$ is not.

Let $A \subset \mathbb{N}$ be a nonempty set of indices i such that $i \not\preccurlyeq_1 n$ and $K_n \sqcup \Delta_i \subset L_m$ for some m.

Let Δ be a copy of $\mathbb{N}$, $X' = X \sqcup \Delta$ a space with the Alexandrov compactification topology $\mathbb{N} \sqcup \Delta$ by the point $0 \in X$. Set

$$K'_m = \begin{cases} K_m \sqcup \Delta, & \text{if } m \succcurlyeq_1 i \text{ for some } i \in A, \\ K_m, & \text{otherwise.} \end{cases}$$

Obviously, $K' = (K'_m)_{m=1}^{\infty} \in F_1X'$. Consider the point $L' = (L'_i)_{i=1}^{\infty} = \varphi X'(K')$.

Let $r\colon X' \to X$ be a retraction such that $r(\Delta) = \{0\}$. Then $F_1r(K') = K$, therefore $F_2r(L') = L$. Hence, every set L'_m equals either L_m or $L_m \sqcup \Delta$ (because the inequality $L'_m \cap \Delta \neq \varnothing$ implies the inclusion $\Delta \subset L'_m$).

If $L'_m \supset \Delta_n$, then $L'_m \supset K_n$ and since $L_m \neq K_n$, we have $L'_m \supset \Delta_i$ for some $i \in A$. Since for an arbitrary number $j \in \mathbb{N}$ the inclusion $K'_j \supset \Delta_i$ implies $K'_j \supset \Delta$, we have $L'_m \supset \Delta$.

Therefore, considering the map $f\colon X' \to X'$,

$$f(x) = y,\ \ f|\Delta = h,\ \ f(t) = t \text{ for other } t \in X',$$

where $h\colon \Delta \to \Delta_n$ is one-to-one and x, y are fixed points of Δ_n, $x \neq y$, one obtains that the point $F_2f(L')$ is invariant with respect to the identical on $X' \setminus \Delta_n$ authomorphisms of X'. However, $F_1f(K')$ is not such a point, because the set $K'_n = K_n$ does not contain Δ. This contradiction completes the proof. □

Proof of Theorem 3.3.4. For every compact Hausdorff space Z denote by Z^* the product $Z \times \mathbb{N}$. Let $X = \mathbb{N}^*$, $Y = X \times \{1,2\}$, αX and αY be the Alexandrov compactifications of X and Y respectively (where 0 is the compactifying point).

For every number $i \in \mathbb{N}$ put

$$N_i = \{j \in \mathbb{N} \mid j \preccurlyeq_1 i\}, \quad M_i = \{j \in \mathbb{N} \mid j \preccurlyeq_2 i\}.$$

Clearly, $N_i \subset N_j$ ($M_i \subset M_j$) if and only if $i \preccurlyeq_1 j$ ($i \preccurlyeq_2 j$). Consider $K_i = N_i^* \sqcup \{0\}$ and $K = (K_i)_{i=1}^{\infty} \in F_1\alpha X$. Then for $L = (L_i)_{i=1}^{\infty} = \varphi\alpha X(K)$, we have by Lemma 3.3.5 (with $\Delta_i = \{i\} \times \mathbb{N}$) that all sets K_m, $m \in \mathbb{N}$, are contained among the sets L_i, $i \in \mathbb{N}$.

Now if

$$P_i = ((L_i \setminus \{0\}) \times \{1\}) \sqcup (M_i^* \times \{2\}) \sqcup \{0\},$$

then $P = (P_i)_{i=1}^{\infty} \in F_2\alpha Y$ (note that this equality admits the formal possibility $L_i = \{0\}$).

Appealing to Lemma 3.3.5 once again (with φ^{-1} and $\Delta_i = P_i \setminus \bigcup\{P_j \mid j \prec_2 i\}$), we obtain that for the point

$$R = (R_i)_{i=1}^{\infty} = (\varphi\alpha Y)^{-1}(P)$$

all P_m, $m \in \mathbb{N}$, belong the family of sets R_i, $i \in \mathbb{N}$.

Let a map $r\colon \alpha Y \to \alpha X$ be defined by the formulae:

$$r(X \times \{2\}) = r(0) = \{0\} \text{ and } r(x,1) = x \text{ for all } x \in X.$$

Then $F_2r(P) = L$, and by the naturality of φ we have $F_1r(R) = K$:

$$\begin{tikzcd}
R = (R_i)_{i=1}^{\infty} \arrow[r, mapsto, "\varphi\alpha Y"] \arrow[d, "F_1 r"'] & P = (P_i)_{i=1}^{\infty} \arrow[d, "F_2 r"] \\
K = (K_i)_{i=1}^{\infty} \arrow[r, mapsto, "\varphi\alpha X"] & L = (L_i)_{i=1}^{\infty}.
\end{tikzcd}$$

Now it is easy to see that the sequence $(L_i)_{i=1}^{\infty}$ is a permutation of the sequence $(K_m)_{m=1}^{\infty}$. Indeed, all K_m, $m \in \mathbb{N}$, is contained among the sets L_i, $i \in \mathbb{N}$, all L_i belong to the family of sets K_m, $m \in \mathbb{N}$, and moreover, if $L_i = L_j$ for $i \neq j$, then for m, n with $R_m = P_i$, $R_n = P_j$, we have $K_m = L_i = L_j = K_n$ and $m = n$, that contradicts $P_i \neq P_j$.

Define a bijection $d\colon \mathbb{N} \to \mathbb{N}$ by the formula $d(i) = j$, whenever $L_j = K_i$. Show that d is an isomorphism of $(\mathbb{N}, \preccurlyeq_1)$ onto $(\mathbb{N}, \preccurlyeq_2)$. If $d(i) \preccurlyeq_2 d(j)$, then $K_i = L_{d(i)} \subset L_{d(j)} = K_j$. Therefore, $N_i \subset N_j$ and $i \preccurlyeq_1 j$.

Since by the diagram above the sequence $(P_i)_{i=1}^{\infty}$ is a permutation of the sequence $(R_i)_{i=1}^{\infty}$, and hence, for every $i \in \mathbb{N}$ the diagram

$$\begin{tikzcd}
R_i \arrow[r, equal] \arrow[d, "\exp r"'] & P_{d(i)} \arrow[d, "\exp r"] \\
K_i \arrow[r, equal] & L_{d(i)}
\end{tikzcd}$$

is commutative, we have

$$R_i = ((K_i \setminus \{0\}) \times \{1\}) \sqcup (M^*_{d(i)} \times \{2\}) \sqcup \{0\}, \quad i \in \mathbb{N}.$$

Therefore if $i \preccurlyeq_1 j$, then $R_i \subset R_j$ and, hence, $M_{d(i)} \subset M_{d(j)}$. Thus, $d(i) \preccurlyeq_2 d(j)$ and the theorem is proved. □

By Theorem 3.3.4, Proposition 3.3.2, and Remark 3.3.3 we have

Corollary 3.3.6. *Monads $\mathbb{F}_{\preccurlyeq_1}$ and $\mathbb{F}_{\preccurlyeq_2}$ are isomorphic if and only if so are the orders $\preccurlyeq_1$ and $\preccurlyeq_2$.* □

Corollary 3.3.7. *The cardinality of skeleton of the normal monads category equals to the continuum.*

Proof. The previous corollary yields the lower estimate. The upper one can be obtained from Propositions 2.6.1 and Theorem 2.6.14. □

3.4. Tensor products

Let $\mathbb{T} = (T, \eta, \mu)$ be a weakly normal monad and $a \in TX$, $b \in TY$. For every $x \in X$ we denote by $i_x \colon Y \to X \times Y$ the map acting by the formula $i_x(y) = (x, y)$, $y \in Y$ and define the map $f_b \colon X \to T(X \times Y)$ by the formula $f_b(x) = Ti_x(b)$, $x \in X$. The map f_b is well-defined, because the map $x \mapsto i_x \colon X \to C(Y, X \times Y)$ is obviously continuous and the functor F is also continuous.

Definition 3.4.1. The element

$$a \otimes b = \mu(X \times Y) \circ Tf_b(a) \in T(X \times Y)$$

is called the *tensor product of a and b.*

Proposition 3.4.2. *1) The map $\otimes \colon TX \times TY \to T(X \times Y)$ is continuous;*
2) $T\operatorname{pr}_1(a \otimes b) = a$, $T\operatorname{pr}_2(a \otimes b) = b$;
3) the operation of tensor product is natural by both arguments;
4) the operation of tensor product is associative in the sense that $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

Proof. Statement 1) follows from the continuity of T and Theorem 2.2.3.
2) Since T preserves singletons, we have

$$T\operatorname{pr}_1 \circ f_b(x) = T(\operatorname{pr}_1 \circ i_x)(b) = \eta X(x).$$

Hence,

$$\begin{aligned} T\operatorname{pr}_1(a \otimes b) &= T\operatorname{pr}_1 \circ \mu(X \times Y) \circ Tf_b(a) = \\ &= \mu X \circ T^2\operatorname{pr}_1 \circ Tf_b(a) = \mu X \circ T\eta X(a) = a. \end{aligned}$$

Since $T\operatorname{pr}_2 \circ f_b(x) = T(\operatorname{pr}_2 \circ i_x) = b$, we have also

$$T\operatorname{pr}_2(a \otimes b) = \mu X \circ T(T\operatorname{pr}_2 \circ f_b)(a) = \mu X \circ \eta TX(b) = b.$$

3) Suppose that the maps $\alpha\colon X \to X'$ and $\beta\colon Y \to Y'$ are given. By primes we denote the corresponding maps used in the definition of the tensor product $\otimes\colon TX' \times TY' \to T(X' \times Y')$. It can be verified directly that $T(\alpha \times \beta) \circ f_b = f'_{T\beta(b)} \circ \alpha$, from which

$$\begin{aligned} T(\alpha \times \beta)(a \otimes b) &= T(\alpha \times \beta) \circ \mu(X \times Y) \circ T f_b(a) = \\ &= \mu(X' \times Y') \circ T^2(\alpha \times \beta) \circ T f_b(a) = \\ &= \mu(X' \times Y') \circ T f'_{T\beta(b)} \circ T\alpha(a) = \\ &= T\alpha(a) \otimes T\beta(b). \end{aligned}$$

4) We introduce the necessary notation. For any $x \in X$ and $y \in Y$ let $\bar{i}_x\colon Y \times Z \to X \times Y \times Z$, $j_y\colon Z \to Y \times Z$, and $k_{(x,y)}\colon Z \to X \times Y \times Z$ be the maps acting according to the formulae

$$\bar{i}_x(y', z') = (x, y', z'), \; j_y(z') = (y, z'), \text{ and } k_{(x,y)}(z') = (x, y, z'),$$

$y' \in Y$, $z' \in Z$. For $s \in T(Y \times Z)$ and $t \in TZ$ define the maps $\bar{f}_s\colon X \to T(X \times Y \times Z)$, $g_t\colon Y \to T(Y \times Z)$, and $h_y\colon X \times Y \to T(X \times Y \times Z)$ by setting

$$\bar{f}_s(x) = T\bar{i}_x(s), \; g_t(y) = Tj_y(t), \; h_t(x, y) = Tk_{(x,y)}(t), \; (x, y) \in X \times Y.$$

Note that

$$(h_c \circ i_x)(y) = h_c(x, y) = Tk_{(x,y)}(c) = T(\bar{i}_x \circ j_y)(c) = T\bar{i}_x \circ g_c(y).$$

Therefore,

$$\begin{aligned} \bar{f}_{b\otimes c}(x) &= T\bar{i}_x(b \otimes c) = T\bar{i}_x \circ \mu(Y \times Z) \circ Tg_c(b) = \\ &= \mu(X \times Y \times Z) \circ T^2\bar{i}_x \circ Tg_c(b) = \mu(X \times Y \times Z) \circ T(T\bar{i}_x \circ g_c)(b) \\ &= \mu(X \times Y \times Z) \circ T(h_c \circ i_x)(b) = \mu(X \times Y \times Z) \circ Th_c \circ f_b(x). \end{aligned}$$

From this we get

$$\begin{aligned} (a \otimes b) \otimes c &= \mu(X \times Y \times Z) \circ Th_c(a \otimes b) = \\ &= \mu(X \times Y \times Z) \circ Th_c \circ \mu(X \times Y) \circ T f_b(a) = \\ &= \mu(X \times Y \times Z) \circ \mu T(X \times Y \times Z) \circ T^2 h_c \circ T f_b(a) = \\ &= \mu(X \times Y \times Z) \circ T(\mu(X \times Y \times Z) \circ Th_c \circ f_b)(a) = \\ &= \mu(X \times Y \times Z) \circ T\bar{f}_{b\otimes c}(a) = a \otimes (b \otimes c). \end{aligned}$$

□

Remark 3.4.3. Condition 3) implies, in particular, that the operation of tensor product determines the natural transformation $\otimes\colon (-)^2T \to T(-)^2$.

Condition 4) of Proposition 3.4.2 allows us to define, for every $a \in TX$, $b \in TY$, $c \in TZ$, the triple tensor product $a\otimes b\otimes c \in T(X\times Y\times Z)$ as $a\otimes(b\otimes c) = (a\otimes b)\otimes c$.

By induction, one can similarly define, for every $a_i \in TX_i$, $i = 1,\dots,n$, the tensor product

$$\otimes_{i=1}^n a_i = a_1\otimes\cdots\otimes a_n \in X_1\times\cdots\times X_n.$$

Given a product $X = \prod_{i\in A} X_i$ in **Comp** and $a_i \in TX_i$, $i \in A$, one can uniquely determine the tensor product $\otimes_{i\in A}a_i \in TX$ by the following condition:

$$T\mathrm{pr}_B(\otimes_{i\in A}a_i) = \otimes_{i\in B}a_i \text{ for every finite } B\subset A$$

($\mathrm{pr}_B\colon \prod_{i\in A}X_i \to \prod_{i\in B}X_i$ is the natural projection).

Examples.

1. For the hyperspace monad $\mathbb{H}$ the tensor product looks as follows: $A\otimes A = A\times B$.
2. For the probability measure monad the tensor product $\mu\otimes\nu$ of $\mu = \sum_{i=1}^k \alpha_i\delta_{x_i}$, $\nu = \sum_{j=1}^l \beta_j\delta_{y_j}$ is

$$\mu\otimes\nu = \sum_{i=1}^k\sum_{j=1}^l \alpha_i\beta_j\delta_{(x_i,y_j)}.$$

One can define a variation of the tensor product in the following manner. Let $\mathbb{T} = (T,\eta,\mu)$ be a weakly normal monad and $a \in TX$, $b \in TY$. For every $y\in Y$ we denote by $j_y\colon X \to X\times Y$ the map acting by the formula $j_y(x) = (x,y)$, $x\in X$ and define the map $g_a\colon Y \to T(X\times Y)$ by the formula $g_a(y) = Tj_y(a)$, $y \in y$. Then let

$$a\otimes' b = \mu(X\times Y)\circ Tg_a(b) \in T(X\times Y).$$

It is easy to see that the operations $\otimes$ and $\otimes'$ coincide for the power monad, the (continuum) hyperspace monad, and for the probability measure monad.

Theorem 3.4.4. *Let $\mathbb{T} = (T,\eta,\mu)$ be a weakly normal monad and $\deg T = n < \infty$. Then $T \cong (-)^n$.*

Proof. Fix $a \in Tn$, $\deg(a) = n$, and for every $i \leq n$ define the map $\sigma_a : C(n,i) \to Ti$ by the formula

$$\sigma_a(f) = Tf(a), \ f \in C(n,i).$$

Show that the map σ_a is bijective. Let $a' \in Ti$. Since $\deg T = n$, Proposition 3.4.2 implies that the restriction of the projection map $\mathrm{pr}_1 : n \times i \to n$ on the subset $\mathrm{supp}(a \otimes a') \subset n \times i$ is a bijection. Thus, the subset $\mathrm{supp}(a \otimes a') \subset n \times i$ is the graph of a map $g : n \to i$. Obviously, $a' = Tg(a) = \sigma_a(g)$. By arbitrarity of a', σ_a is onto.

To check the injectivity of σ_a suppose that $\sigma_a(f_1) = \sigma_a(f_2)$ for some $f_1, f_2 \in C(n,i)$. Let $c = a \otimes a \in T(n \times n)$. By Proposition 3.4.2,

$$T(1_n \times f_1)(c) = a \otimes Tf_1(a) = a \otimes Tf_2(a) = T(1_n \times f_2)(c)$$

and therefore, $(1_n \times f_1)(n) = (1_n \times f_2)(n)$, i.e., $f_1 = f_2$.

Define the natural transformation $\psi : T|\mathrm{K}_n \to (-)^n|\mathrm{K}_n$ by the following manner. Let $\psi_n(a) = (0, 1, \ldots, n-1) \in n^n$. For $a' \in Ti$, where $i \leq n$, it is shown that there exists a unique map $f \in C(n,i)$ such that $a' = Tf(a)$. Let

$$\psi i(a') = f^n(0, 1, \ldots, n-1) \in i^n.$$

Note that for a map $g : i \to j$ we have

$$\psi j \circ Tg(a) = \psi j \circ T(g \circ f)(a) = (g \circ f)^n(0, 1, \ldots, n-1) = g^n \circ \psi i(a'),$$

i.e., the natural transformation ψ is well-defined. Obviously, ψ is a functorial isomorphism, and, by Proposition 2.6.7, ψ can be extended to a functorial isomorphism $\bar{\psi} : T \to (-)^n$. □

Theorem 3.4.5. *Let $\mathbb{T} = (T, \eta, \mu)$ be a weakly normal monad, where T is a functor with finite supports. Then $\mathbb{T}$ is the n-power monad for some n.*

Proof. Show that $\deg T < \infty$. Suppose the contrary. Consider an inverse sequence

$$Q_1 \xleftarrow{p_1} Q_1 \times Q_2 \xleftarrow{p_2} Q_1 \times Q_2 \times Q_3 \xleftarrow{p_3} \cdots, \tag{3.8}$$

where Q_i are copies of the Hilbert cube and p_i the projections on respective factors. Let $(\prod_{i=1}^{\infty} Q_i, \pi_i)$ be the inverse limit of (3.8). Choose

$a_i \in TQ$ with $\deg(a_i) \geq i$. Then a sequence $(a_1 \otimes \cdots \otimes a_i)_{i=1}^{\infty}$ forms a point $a \in T(\prod_{i=1}^{\infty} Q_i)$. Clearly, $\deg(a) = \infty$. This contradiction and Theorem 3.4.4 yield the equality $T \cong (-)^n$ for some $n < \infty$. □

Theorem 3.4.6. *Let $\mathbb{T} = (T, \eta, \mu)$ be a weakly normal monad, where T is a functor with countable kernels. Then $T \cong (-)^\alpha$, where $1 \leq \alpha \leq \omega$.*

Proof. First, show that there exists an element $a \in T\beta\omega$ with $\ker(a) = \omega$, satisfying the following property:

(∗) for every $b \in TX$ there exists a map $f : \omega \to X$ such that $Tf(a) = b$.

To this end, for every $c \in T\beta\omega$ with $\ker(c) = \omega$, fix a copy $\beta\omega^{(c)}$ of $\beta\omega$. Let

$$a' = \otimes\{c \mid c \in T(\beta\omega^{(c)}),\ \ker(c) \subset \omega \subset \beta\omega^{(c)}\}.$$

Since T is a functor with countable kernels, we see that $|\ker(a)| = \omega$ and, consequently, there exists a surjection $f : \omega \to \ker(a')$ and $a \in T\beta\omega$ such that $T\beta f(a) = a'$.

Now, show that this element can be chosen so that it satisfy the following condition:

(∗∗) for every map $g : Y \to \beta\omega$ and every $c \in TY$ such that $Tg(c) = a$ there exists a map $h : \omega \to Y$ such that $gh = 1_\omega$, $Th(a) = c$.

Indeed, suppose the contrary. We can construct, by transfinite induction, elements $a_\alpha \in T\beta\omega^{(\alpha)}$, where $\omega^{(\alpha)}$ is a copy of ω, and, for every $\alpha > \gamma$, maps $p_{\alpha\gamma} : \omega^{(\alpha)} \to \omega^{(\gamma)}$ such that (α, γ are countable ordinals):

1) $a_\gamma = T\beta p_{\alpha\gamma}(a_\alpha)$;

2) $p_{\alpha\gamma}$ is a surjection but not a bijection.

The construction starts with $a_0 = a \in T\beta\omega = T\beta\omega^{(0)}$. If $\alpha = \beta + 1$, we can construct a_α and $p_{\alpha\gamma}$ using the fact that condition (∗∗) does not hold (with $a = a_\gamma$ $\omega = \omega^{(\gamma)}$).

If α is a limit ordinal, let $a' = \otimes_{\gamma<\alpha} a_\gamma \in T(\prod_{\gamma<\alpha} \beta\omega^{(\gamma)})$. There exists a surjection $q : \omega^{(\alpha)} \to \ker(a')$ and an element $a_\alpha \in T\beta\omega^{(\alpha)}$ such that $T\beta q(a_\alpha) = a'$. We let $p_{\alpha\gamma} = \pi_\gamma q$, where $\pi_\gamma : \prod_{\gamma'<\alpha} \omega^{(\gamma')} \to \omega^{(\gamma)}$ is the projection.

Now let $x = \otimes_{\alpha<\omega_1} a_\alpha$. To obtain a contradiction, it is sufficient to show that $\ker(x)$ is uncountable. By transfinite induction one can easily define, for every $\alpha \leq \gamma < \omega_1$, embeddings $s_{\alpha\gamma} : \omega^{(\alpha)} \to \omega^{(\gamma)}$ such that $p_{\alpha\gamma} \circ s_{\alpha\gamma} = 1_{\omega^{(\alpha)}}$ and $s_{\alpha\gamma} \circ s_{\gamma\xi} = s_{\alpha\xi}$, whenever $\alpha \leq \gamma \leq \xi$. To simplify notation, we identify every $\omega^{(\alpha)}$ with its image under the map $s_{\alpha\gamma}$; then

$p_{\alpha\gamma}$ become retractions. Then for every $\alpha < \omega_1$ the set $\ker(x)$ retracts onto $\ker \prod_{\gamma<\omega_1} a'_{\alpha\gamma}$, where

$$a'_{\alpha\gamma} = \begin{cases} Tp_{\alpha\gamma}(a_\alpha), & \text{if } \gamma \leq \alpha, \\ a_\alpha, & \text{otherwise.} \end{cases}$$

This obviously implies uncountability of $\ker(x)$.

Let $X \in |\mathbf{Comp}|$. Define a map $\xi X : TX \to X^\omega$ in the following manner. Fix $a \in T\beta\omega$ with $\ker(a) = \omega$ satisfying property $(*)$. Given $b \in TX$, we see that the set $\ker(a \otimes b)$ is the graph of a map $f : \omega \to X$. Then, obviously, $b = T\beta f(a)$. We put $\xi X(b) = f \in X^\omega \equiv C(\omega, X)$. Because of lower semicontinuity of the map supp, we see that the map ξX is continuous. It is easy to see that $\xi = (\xi X) : (-)^\omega \to T$ is a natural transformation.

If $b \neq b'$, then $\xi X(b) \neq \xi X(b')$, i. e. the map ξX is injective. Prove that ξX is an onto map. Let $f \in C(\omega, X)$, then

$$T(1_{\beta\omega} \times \beta f)(a \otimes a) = (a \otimes T\beta f(a)),$$

whence $\xi X(T\beta f(a)) = f$. □

3.4.1. Extension of monad structure onto the category Tych

Let $\mathbb{T} = (T, \eta, \mu)$ be a normal monad on the category of compacta. Suppose that the functor T has continuous supports. Then by Theorem 2.7.7 and Proposition 2.7.5 the natural transformation $\mu : T^2 \to T$ has a unique extension $\mu_\beta : (T^2)_\beta = (T_\beta)^2 \to T_\beta$ onto the category **Tych**.

Theorem 3.4.7. *The triple $\mathbb{T}_\beta = (T_\beta, \eta_\beta, \mu_\beta)$ forms a monad on* **Tych**.

Proof. Let X be a Tychonov space, $a \in T_\beta X$, and $\text{supp}(a) = A$. Then

$$\mu_\beta X \circ \eta_\beta T_\beta X(a) = \mu A \circ \eta T A(a) = a.$$

If $b \in (T_\beta)^3 X$, then by one can obtain that $b \in T^3 B \subset (T_\beta)^3 X$ for some compact Hausdorff space $B \subset X$. Finally, by naturality of μ we have

$$\mu_\beta X \circ \mu_\beta T_\beta X(b) = \mu B \circ \mu T B(b) = \mu B \circ T\mu B(b) = \mu_\beta X \circ T_\beta \mu_\beta X(b).$$ □

The following example shows that one cannot avoid the condition of continuity of supports in Theorem 3.4.7.

Example. Consider the probability measure monad $\mathbb{P}$. If

$$M = \sum_{i=1}^{\infty} \delta_{1/i\delta_1 + (i-1)/i\delta_i} \in P^2\omega,$$

then $M \in P_\beta P_\beta \omega$, but $\psi\omega(M) \notin P_\beta\omega$.

Let $\mathbb{T} = (T, \eta, \mu)$ be a (weakly, almost) normal monad in **Comp**. The natural transformations η and μ uniquely determine the natural transformations $\eta_\infty \colon 1_{\mathbf{Comp}^\infty} \to T_\infty$, $\mu_\infty \colon (T_\infty)^2 \to T_\infty$.

Proposition 3.4.8. *The triple $\mathbb{T}_\infty = (T_\infty, \eta_\infty, \mu_\infty)$ is a monad in the category* $\mathbf{Comp}^\infty$.

Proof. Straightforward. □

Some monads in the category $\mathbf{Comp}^\infty$ are generated by functors that are countable direct limits of functors of finite degree.

Examples. The functor of free topological group $FG \colon \mathbf{Comp}^\infty \to \mathbf{Comp}^\infty$ is described as follows: In Section 2.5 the functors FG_n of words of length $\leq n$ are defined. Let $FGX = \varinjlim FG_nX$. The elements of FGX are the finite words of the form $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$, where $\varepsilon_i = \pm 1$. The natural transformation $\mu \colon (FG)^2 \to FG$ is the "removing parentheses". If we restrict ourselves with the subsets of words in which all ε_i are $= 1$, we obtain the topological free semigroup submonad of the free topological group monad.

Similarly, the monad of free Abelian topological group (AG, η, μ) can be defined.

3.4.2. Monad generated by the functor C_pC_p and its submonads

A series of examples of monads in the categories **Tych** and $\mathbf{Comp}^\infty$ can be obtained as submonads of the monad generated by the functor C_pC_p.

For a Tychonov space X we denote by C_pX the space of real-valued functions on X endowed by the topology of pointwise convergence. This construction determines a contravariant functor in **Tych**: for a map $f \colon X \to Y$ we have $C_pf(\varphi) = \varphi \circ f$, $\varphi \in C_pY$.

Define the natural transformation $\eta \colon 1_{\mathbf{Tych}} \to C_pC_p = C_p^2$ by the condition: $\eta X(x)(\varphi) = \varphi(x)$, where $x \in X$, $\varphi \in C_pX$.

Lemma 3.4.9. $C_p\eta \circ \eta C_p = 1_{C_p}$.

Proof. For every $x \in X$ and $\varphi \in C_pX$ we have

$$\begin{aligned} C_p\eta X \circ \eta C_pX(\varphi)(x) =& C_p\eta X(\eta C_pX(\varphi))(x) = \eta C_pX(\varphi)(\eta X(x)) \\ =& = \eta X(x)(\varphi) = \varphi(x). \end{aligned}$$

□

Using Lemma 3.4.9 and Proposition 1.2.3 we see that the functor $T_p = C_p^2$ determines a monad on the category **Tych**.

Consider some submonads of the monad (C_p^2, η, μ). First note that the space C_pC_pX is a topological algebra with respect to the linear operations and pointwise multiplication. Let

$$\begin{aligned} L_pX =& \{\sum_{i=1}^{k} \lambda_i \eta X(x_i) \mid \lambda_i \in \mathbb{R},\ x_i \in X, \quad k \in \mathbb{N}\}' \\ G_pX =& \{\sum_{i=1}^{k} \eta X(x_i) \mid x_i \in X, \quad k \in \mathbb{N}\}, \\ M_pX =& \{\prod_{i=1}^{k} \eta X(x_i) \mid x_i \in X, \quad k \in \mathbb{N}\}, \\ A_pX =& \{\sum_{i=1}^{k} \lambda_i \prod_{j_i=1}^{l_i} \eta X(x_{ij_i}) \mid \lambda_i \in \mathbb{R},\ x_{ij_i} \in X, \quad k, l_i \in \mathbb{N}\} \end{aligned}$$

It is easy to see that L_p, G_p, M_p i A_p are subfunctors in C_pC_p.

Proposition 3.4.10. *The subfunctors L_p, G_p, M_p i A_p generate submonads in (C_pC_p, η, μ).*

Proof. Because of similarity, we consider only the case of the functor M_p. Let $\Phi \in M_pM_pX$, $X \in$ **Tych**, then $\Phi = \prod_{i=1}^{k} \eta M_pX(F_i)$, $F_i \in M_pX$, and $F_i = \prod_{j_i=1}^{l_i} \eta X(x_{j_i})$, $x_{j_i} \in X$. Then for every $\varphi \in C_pX$ we have

$$\begin{aligned} \mu X(\Phi)(\varphi) =& C_p\eta C_pX(\Phi)(\varphi) = (\Phi \circ \eta C_pX)(\varphi) \\ =& \left(\prod_{i=1}^{k} \eta M_pX(F_i)\right)(\eta C_pX(\varphi)) = \prod_{i=1}^{k}(\eta M_pX(F_i)(\eta C_pX(\varphi))) \\ =& \prod_{i=1}^{k} \eta C_pX(\varphi)(F_i) = \prod_{i=1}^{k} F_i(\varphi) = \prod_{i=1}^{k}\prod_{j_i=1}^{l_i} \eta X(x_{j_i})(\varphi), \end{aligned}$$

i. e. $\mu X(\Phi) = \prod_{i=1}^{k} \prod_{j_i=1}^{l_i} \eta X(x_{j_i}) \in M_pX$. □

Note that the functors L_p, G_p, M_p i A_p can be considered as endofunctors in $\mathbf{Comp}^\infty$. Thus, they determine monads in $\mathbf{Comp}^\infty$. Besides, remark that these functors can be represented as countable direct limits of functors of finite degree.

Exercises

1. Let F be a subfunctor of $1_{\mathbf{Comp}} \times \exp$, $FX = \{(x, A) \in X \times \exp X \mid x \in A\}$. Define natural transformations $\delta\colon F \to 1_{\mathbf{Comp}}$, $\nu\colon F \to F^2$ by the formulae:
$$\delta X(x, A) = x,\ \nu X(x, A) = ((x, A), \{(y, A) \mid y \in A\}).$$
Prove that the triple (F, δ, ν) is a comonad in **Comp**, i. e. $F\nu \circ \nu = \nu F \circ \nu$, $\delta F \circ \nu = F\delta \circ \nu = 1_F$.
2. Prove the counterpart of Proposition 3.4.2 for arbitrary tensor products.
3. Show that the operations $\otimes$ and $\otimes'$ do not coincide for the inclusion hyperspace monad, the superextension monad, and the monads generated by the functors N_k.
4. Prove the counterpart of Proposition 3.4.2 for the tensor product $\otimes'$.
5. Prove that for every (weakly, almost) normal monad $\mathbb{T} = (T, \eta, \mu)$ and every countable ordinal α the α-th iteration T^α of T is a (weakly, almost) normal functor.
6. Denote by **CSG** the category of compact Hausdorff semigroups and continuous homomorphisms. Let $\mathbb{T} = (T, \eta, \mu)$ be (weakly, almost) normal monad. For every $(S, m) \in |\mathbf{CSG}|$ define a map $\bar{m}\colon TS \times TS \to TS$, $\bar{m}(a, b) = Tm(a \otimes b)$. Show that the correspondence $(S, m) \mapsto (TS, \bar{m})$ determines a lifting of the functor T onto the category **CSG** (with respect to the forgetful functor $U\colon \mathbf{CSG} \to \mathbf{Comp}$).
7. Suppose that $\mu \in PX$ is an atomic measure, $X \in |\mathbf{Comp}|$. Is the space $(\psi X)^{-1}(\mu)$ metrizable?
8. Denote by $\exp^{cf}\colon \mathbf{Tych} \to \mathbf{Tych}$ the subfunctor of $\exp \equiv \exp_\beta$ defined as follows:
$$\exp^{cf} = \{A \in \exp X \mid A \text{ consists of a finite number of connected components}\}.$$
Prove that $\exp^{cf}$ determines a submonad of the hyperspace monad in **Tych**.
9. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in **Comp** and the functor $T^\infty\colon \mathbf{Comp} \to \mathbf{Tych}$ is defined as follows: $T^\infty = \varinjlim\{T^i, \eta T^i\}$. Prove that T^∞ is a right $\mathbb{T}$-functor.

Problems

1. Are there the functors $\exp^\alpha$ mutually nonisomorphic?
2. Let F be a (weakly, almost) normal functor. The *free monad* generated by F is a (weakly, almost) normal monad $\mathbb{T} = (T, \eta, \mu)$ with a natural transformation $\alpha\colon F \to T$ satisfying the condition: for every (weakly, almost) normal monad $\mathbb{T}' = (T', \eta', \mu')$ and every natural transformation $\beta\colon F \to T'$ there exists a monad morphism $\gamma\colon T \to T'$ such that $\beta = \gamma \circ \alpha$.
   1) Show that the identity monad is the free monad generated by the identity functor.
   2) Is there a free monad generated by the hyperspace functor?
3. Let $\mathbb{T} = (T, \eta', \mu')$ be a monad in **Comp** (respectively $\mathbf{Comp}^\infty$). We say that $\mathbb{T}$ *embeds* into the monad (C_pC_p, η, μ) if there is a natural embedding $\alpha\colon T \to C_pC_p$ such that $\alpha \circ \eta' = \eta$ and $\mu \circ C_pC_p\alpha \circ \alpha T = \alpha \circ \mu'$. What (weakly, almost) normal monads embed in (C_pC_p, η, μ)?

3.5. Extension of functors onto the Kleisli category

This section is devoted to the problem of extension of (weakly, almost) normal functors onto the Kleisli categories of (weakly, almost) normal monads. Note that some results in this direction are of general character (see Section 1.2.3).

Theorem 3.5.1. *Let $\mathbb{T} = (T, \eta, \mu)$ be a projective monad and $F\colon \mathcal{C} \to \mathcal{C}$ an endofunctor. Then there exists an extension of F onto $\mathcal{C}_{\mathbb{T}}$.*

Proof. Let $\pi\colon T \to 1_{\mathcal{C}}$ be a projection. It is easy to see that the natural transformation $\xi = \eta F \circ F\pi$ satisfies Theorem 1.2.8. □

Theorem 3.5.2. *Let $\mathbb{T} = (T, \eta, \mu)$ be a normal monad on the category* **Comp** *and the functor* $\exp_3$ *extend onto* $\mathbf{Comp}_{\mathbb{T}}$. *Then $\mathbb{T}$ is projective.*

Proof. Let $c = \{0, 1, 2\} \in \exp_3(3)$. For every X denote by $f_i\colon X \to 3\times X$ the following map:

$$f_i(x) = (i, x) \in 3 \times X.$$

For every $a \in TX$ put

$$c \oplus a = \{Tf_i(a) \mid i \in 3\} \in \exp_3 T(3 \times X).$$

Denote by $\xi\colon \exp_3 T \to T\exp_3$ a natural transformation corresponding to the extension of functor $\exp_3$ onto the category $\mathbf{Comp}_{\mathbb{T}}$ (see Theorem 1.2.8).

Let $\pi_1 X$ ($\pi_2 X$) be the projection of $3\times X$ to the first (second) factor.

Lemma 3.5.3. *For every $a \in TX$ we have $T\exp_3 \pi_2 X \circ \xi(3\times X)(c\oplus a) \in TsX(TX)$ (here s is a natural transformation* $\mathrm{Id} \to \exp_3$*).*

Proof. By naturality of ξ

$$\begin{aligned} T\exp_3 \pi_1 X \circ \xi(3 \times X)(c \oplus a) &= \xi 3 \circ \exp_3 T\pi_1 X(c \oplus a) = \\ &= \xi 3(\{\eta X(0), \eta X(1), \eta X(2)\}) = \xi 3 \circ \exp_3 \eta 3(c) = \eta \exp_3 3(c), \end{aligned}$$

and, therefore, for every $A \in \mathrm{supp}_T(\xi(3\times X)(c\oplus))$ we have $\pi_1 X(A) = 3$.

Show that every set $A \in \mathrm{supp}_T(\xi(3 \times X)(c\oplus))$ equals to $3 \times \{x\}$ for some $x \in X$. Indeed, otherwise, we can suppose that $(1, x_1), (2, x_2) \in A$

for some $x_1, x_2 \in X$, $x_1 \neq x_2$. Denote by r a retraction of the set 3 onto $2 \subset 3$ such that $r(2) = 1$. Then the element $\exp_3 T(r \times 1_X)(c \oplus a) \in \exp_3 T(2 \times X)$ is invariant with respect to the authomorphisms of $2 \times X$, preserving fibers of the projection onto the second factor. Thus, by naturality of ξ, we see that $\xi(2 \times X) \circ \exp_3 T(r \times 1_X)(c \oplus a)$ also satisfies this property. But this contradicts with the following: the set $B \cap (\{0\} \times X)$ is a singleton for every $B \in \mathrm{supp}_T(\xi(2 \times X) \circ \exp_3 T(r \times 1_X)(c \oplus a))$,

$$(r \times 1_X)(A) \in \mathrm{supp}_T(\xi(2 \times X) \circ \exp_3 T(r \times 1_X)(c \oplus a)),$$

and

$$(r \times 1_X)(A) \cap (\{0\} \times X) = \{(1, x_1), (1, x_2)\}.$$

Hence, for every $A \in \mathrm{supp}_T(\xi(3 \times X)(c \oplus a))$ the set $\pi_2 X(A)$ is a singleton, and we obtain the statement of lemma. $\square$

This lemma allows us to define a map $\psi X : TX \to TX$, putting

$$\psi X(a) = (TsX)^{-1} \circ T \exp_3 \pi_2 X \circ \xi(3 \times X)(c \oplus a), \quad a \in TX.$$

By naturality of ξ, we have

$$\psi X(a) = (TsX)^{-1} \circ \xi X \circ \exp_3 T\pi_2 X(c \oplus a) = (TsX)^{-1} \circ \xi X \circ sTX(a).$$

Therefore, $\psi = (\psi X) : T \to T$ is a natural transformation.

Lemma 3.5.4. *The natural transformation ψ is a morphism of the monad $\mathbb{T}$ into itself.*

Proof. Let $x \in X$. Then

$$\begin{aligned}\psi X \circ \eta X(x) &= (TsX)^{-1} \circ \xi X \circ sTX \circ \eta X(x) \\ &= (TsX)^{-1} \circ \xi X \circ \exp_3 \eta X \circ sX(x) \\ &= (TsX)^{-1} \circ \eta \exp_3 X \circ sX(x) = \eta X \circ (sX)^{-1} \circ sX = \eta X,\end{aligned}$$

i.e., $\psi \circ \eta = \eta$. If $\mathcal{A} \in T^2 X$, then

$$\begin{aligned}&\mu X \circ T\psi X \circ \psi TX(\mathcal{A}) \\ &= \mu X \circ (T^2 sX)^{-1} \circ T\xi X \circ TsTX \circ (TsTX)^{-1} \circ \xi TX \circ sT^2 X(\mathcal{A}) \\ &= \mu X \circ (T^2 sX)^{-1} \circ T\xi X \circ \xi TX \circ sT^2 X(\mathcal{A}) \\ &= (TsX)^{-1} \circ \mu \exp_3 \circ T\xi X \circ \xi TX \circ sT^2 X(\mathcal{A}) = \\ &= (TsX)^{-1} \circ \xi X \circ \exp_3 \mu X \circ T\xi X \circ sT^2 X(\mathcal{A}) \\ &= (TsX)^{-1} \circ \xi X \circ sTX \circ \mu X(\mathcal{A}) = \psi X \circ \mu X(\mathcal{A}),\end{aligned}$$

i.e., $\psi \circ \mu = \mu \circ T\psi \circ \psi T$. $\square$

Define a normal subfunctor S of the functor T by the formula $SX = \psi X(TX) \subset TX$.

Show that S is profinitely power. Let X be a finite Hausdorff space, $b \in SX$ and $a \in TX$ satisfy $\psi X(a) = b$. Without restricting generality we can suppose that $\mathrm{supp}(b) = X$.

It is easy to prove that for every $f_1, f_2 \in C(X, X)$, the inequality $f_1 \neq f_2$ implies $Tf_1(b) \neq Tf_2(b)$. Indeed, otherwise, define maps $h, q\colon 3 \times X \to 3 \times X$ by the following formulae:

$$h(0,x) = q(0,x) = (0,x),\ h(i,x) = \bigl(i, f_i(x)\bigr),\ i = 1,2;$$
$$q(1,x) = \bigl(1, f_2(x)\bigr),\ q(2,x) = \bigl(2, f_1(x)\bigr).$$

Then

$$\exp_3 T(q \circ h)(c \oplus a) = \exp_3 Th(c \oplus a)$$

and

$$\xi(3 \times X) \circ \exp_3 T(q \circ h)(c \oplus a) = \xi(3 \times X) \circ \exp_3 Th(c \oplus a),$$

i.e.,

$$T \exp_3(q \circ h) \circ \xi(3 \times X)(c \oplus a) = T \exp_3 h \circ \xi(3 \times X)(c \oplus a).$$

Now let $x_0 \in X$ be a point with $f_1(x_0) \neq f_2(x_0)$. By the proof of Lemma 3.5.3, we have

$$\bigl\{(0,x_0), \bigl(1, f_1(x_0)\bigr), \bigl(2, f_2(x_0)\bigr)\bigr\} \in \mathrm{supp}_T\bigl(T \exp_3 h \circ \xi(3 \times X)(c \oplus a)\bigr) \setminus \mathrm{supp}_T\bigl(T \exp_3(q \circ h) \circ \xi(3 \times X)(c \oplus a)\bigr).$$

Contradiction.

Clearly, the natural transformations η and μ determine natural transformations $\eta'\colon 1_{\mathbf{Comp}} \to S$, $\mu'\colon S^2 \to S$ such that the triple $\mathbb{S} = (S, \eta', \mu')$ is a monad and $\psi\colon \mathbb{T} \to \mathbb{S}$ is a monad morphism. $\square$

Corollary 3.5.5 (a criterium of projectivity). *Let $\mathbb{T}$ be a normal monad. Then T is projective if and only if the functor $\exp_3$ extends onto the Kleisli category of $\mathbb{T}$.*

Theorem 3.5.6. *A normal functor F of finite degree $n \geq 1$ can be extended onto the category* $\mathbf{Comp}_{\mathbb{H}}$ *if and only if F is isomorphic to SP^n_G for some subgroup $G \subset S_n$. There is a unique extension of SP^n_G onto* $\mathbf{Comp}_{\mathbb{H}}$.

Proof. Sufficiency. Define the map $d_G X \colon SP^n_G \exp X \to \exp SP^n_G X$ by the formula

$$d_G X([A_0, \dots, A_{n-1}]_G) = \{[a_0, \dots, a_{n-1}]_G \mid a_i \in A_i,\ i \in n\}.$$

For the trivial group G instead of $d_G X$ we use the map $d_0 X : (\exp X)^n \to \exp(X^n)$,

$$d_0 X(A_0, \dots, A_{n-1}) = A_0 \times \dots \times A_{n-1}.$$

Evidently, the map $d_0 X$ is continuous. Since

$$d_G X = \exp \pi_G \circ u X^n \circ \exp d_0 X \circ (\pi_G \exp X)^{-1}$$

and the map $\pi_G \exp X$ is open, we see that the map $d_G X$ is continuous for every subgroup $G \subset S_n$. It is easy to see that $d_G = (d_G X)$ is a natural transformation.

We have

$$d_G X \circ SP^n_G s X([x_0, \dots, x_{n-1}]_G) = d_G X([\{x_0\}, \dots, \{x_{n-1}\}]_G) =$$
$$= \{[x_0, \dots, x_{n-1}]_G\} = s SP^n_G X([x_0, \dots, x_{n-1}]_G),$$

i.e., $d_G \circ SP^n_G s = s SP^n_G$.

Show that

$$u SP^n_G \circ \exp d_G \circ d_G \exp = d_G \circ SP^n_G u.$$

Let $[\mathcal{A}_0, \dots, \mathcal{A}_{n-1}]_G \in SP^n_G \exp^2 X$. Then

$$\begin{aligned}
&u SP^n_G X \circ \exp d_G X \circ d_G X([\mathcal{A}_0, \dots, \mathcal{A}_{n-1}]_G)\\
&\quad= u SP^n_G X \circ \exp d_G X(\{[A_0, \dots, A_{n-1}]_G\} \mid A_i \in \mathcal{A}_i,\ i \in n\})\\
&\quad= u SP^n_G X(\{\{[a_0, \dots, a_{n-1}]_G \mid a_j \in A_j,\ j \in n\} \mid A_i \in \mathcal{A}_i,\ i \in n\})\\
&\quad= \{[a_0, \dots, a_{n-1}]_G \mid a_i \in uX(\mathcal{A}_i),\ i \in n\}\\
&\quad= d_G X([uX(\mathcal{A}_0), \dots, uX(\mathcal{A}_{n-1})]_G)\\
&\quad= d_G X \circ SP^n_G u X([\mathcal{A}_0, \dots, \mathcal{A}_{n-1}]_G).
\end{aligned}$$

By Theorem 1.2.8, the functor SP^n_G can be extended onto the category $\mathbf{Comp}_{\mathbb{H}}$.

Now we are able to prove that this extension is unique. Suppose that

$$d'_G\colon SP^n_G \exp \to \exp SP^n_G$$

is a natural transformation such that $d'_G \circ SP^n_G s = sSP^n_G$. In order to prove the uniqueness it is sufficient to show that $d'_G = d_G$.

Let $m \geq 1$. Define the map $g_m\colon n \to \exp(n \times m)$ by the formula $g_m(i) = \{(i,j) \mid j \in m\}$.

Let $a \in SP^n_G(n)$, $\deg(a) = n$. Then

$$d'_G(n \times m) \circ SP^n_G g_m(a) \subset SP^n_G(n \times m).$$

For every $b \in d'_G(n \times m) \circ SP^n_G g_m(a)$ we have

$$\begin{aligned}
SP^n_G \operatorname{pr}_1(b) &\in \exp SP^n_G \operatorname{pr}_1 \circ d'_G(n \times m) \circ SP^n_G g_m(a)\\
&= d'_G n \circ SP^n_G \exp \operatorname{pr}_1 \circ SP^n_G g_m(a) = d'_G n \circ SP^n_G(\exp \operatorname{pr}_1 \circ g_m)(a)\\
&= d'_G n \circ SP^n_G sn(a) = sSP^n_G(n)(a).
\end{aligned}$$

Necessity. Let F be a normal functor, $\deg F = n \geq 1$. Suppose that F extends onto the Kleisli category of the hyperspace monad. By Theorem 1.2.8, there exists a natural transformation $d\colon F\exp \to \exp F$ with $d \circ Fs = sF$.

Let $a \in Fn$, $\deg(a) = n$, and $g_n\colon \to \exp(n \times n)$ be as above. Thinking similarly to the previous paragraphs, we obtain that

$$d(n \times n) \circ Fg_n(a) = \{Ff(a) \mid f \in C(n,n),\ \operatorname{pr}_1 \circ f = \mathrm{id}_n\}.$$

Denote by $\eta\colon \mathrm{Id} \to F$ a unique natural tranformation. Note that for every $a' \in Fn$ we have

$$F\exp \operatorname{pr}_2 \circ Fg_n(a') = F(\exp \operatorname{pr}_2 \circ g_n)(a') = \eta \exp n(n) \in F\exp n$$

and

$$\begin{aligned}
dn \circ F\exp \operatorname{pr}_2 \circ Fg_n(a') &= dn \circ F\exp \operatorname{pr}_2 \circ Fg_n(a) =\\
&= \exp F\operatorname{pr}_2 \circ d(n \times n) \circ Fg_n(a)\\
&= \exp F\operatorname{pr}_2(\{Ff(a) \mid f \in C(n, n \times n),\ \operatorname{pr}_1 \circ f = \mathrm{id}_n\})\\
&= \{F(\operatorname{pr}_1 \circ f)(a) \mid f \in C(n, n \times n),\ \operatorname{pr}_1 \circ f = \mathrm{id}_n\}\\
&= \{Ff'(a) \mid f' \in C(n,n)\}.
\end{aligned}$$

Hence, we obtain that

$$a \in dn \circ F \exp \mathrm{pr}_2 \circ Fg_n(a') = \exp F\mathrm{pr}_2 \circ d(n \times n) \circ Fg_n(a').$$

There exist $b \in d(n \times n) \circ Fg_n(a')$ such that $F\mathrm{pr}_2(b) = a$. Obviously, then $\deg(b) = n$ and

$$\begin{aligned} F\mathrm{pr}_1(b) \in \exp F\mathrm{pr}_1 \circ d(n \times n) \circ Fg_n(a') = dn \circ F \exp \mathrm{pr}_1 \circ Fg_n(a') = \\ = dn \circ F(\exp \mathrm{pr}_1 \circ g_n)(a') = dn \circ Fsn(a') = sFn(a') = \{a'\}, \end{aligned}$$

i.e., $F\mathrm{pr}_2(b) = a'$. Hence, there exists a map $f' \in C(n, n)$ with $Ff'(a) = a'$. Then maps $\xi X \colon X^n = C(n, X) \to FX$, $\xi X(f) = Ff(a)$, $f \in C(n, X)$, are onto and, moreover, thay determine a natural transformation $\xi \colon (-)^n \to F$.

Now let $a' \in Fn$ and $f \in C(n, n)$ satisfy $Ff(a) = a'$. Then

$$\begin{aligned} d(n \times n) \circ Fg_n(a') &= d(n \times n) \circ F(g_n \circ f)(a) \\ &= d(n \times n) F(\exp(j \times \mathrm{id}_n) \circ g_n)(a) \\ &= \exp F(f \times \mathrm{id}_n) \circ d(n \times n) \circ Fg_n(a) \\ &= \exp F(f \times \mathrm{id}_n)(\{Ff'(a) \mid f' \in C(n, n \times n),\ \mathrm{pr}_1 \circ f' = \mathrm{id}_n\}) \\ &= \{Ff''(a) \mid f'' \in C(n, n \times n),\ \mathrm{pr}_1 \circ f'' = f\}. \end{aligned}$$

Show that the diagram

$$\begin{tikzcd} n^n \arrow[r, "q^n"] \arrow[d, "\xi n"'] & n^n \arrow[d, "\xi n"] \\ Fn \arrow[r, "Fq"] & Fn \end{tikzcd} \tag{3.9}$$

is bicommutative for each $q \in C(n, n)$. For this show firstly the following: for each $g \in n^n$ with $\xi n(g) = Fq(a)$ there exists a bijection $h \colon n \to n$ such that $q^n(h) = q \circ h = g$ and $\xi n(h) = Fh(a) = a$. Indeed, we have

$$F(g, \mathrm{id}_n)(a) \in d(n \times n) \circ Fg_n \circ Fg(a).$$

Since $Fq(a) = \xi n(g) = Fg(a)$, we have moreover

$$F(g, \mathrm{id}_n)(a) \in d(n \times n) \circ Fg_n \circ Fq(a).$$

Thus, $(g, \mathrm{id}_n) = (q, h^{-1}) \circ h$ for some bijection $h \in C(n, n)$ such that $F(g, \mathrm{id}_n)(a) = F(q, h^{-1})(a)$. Clearly, $Fh(a) = a$ and

$$g = \mathrm{pr}_1 \circ (g, \mathrm{id}_n) = \mathrm{pr}_1 \circ (q, h^{-1}) \circ h = q \circ h.$$

Now let points $c, c' \in Fn$ and $g \in n^n$ be such that $Fq(c) = \xi n(g) = c'$. Then there exists a map $r \in C(n, n)$ with $c = Fr(a)$. Choose $h' \in n^n$, satisfying $\xi n(h') = a$, $(qr)^n(h') = g$, and put $h'' = r^n(h)$. We obtain that

$$\xi n(h'') = Fh''(a) = F(r \circ h)(a) = Fr(a) = c$$

and

$$q^n(h'') = (q \circ r)^n(h') = g,$$

i.e., diagram (3.9) is bicommutative. By Lemma 2.10.23, F is an open quotient-functor of $(-)^n$. Therefore, by Theorem 2.10.21, $F \cong SP^n_G$ for some subgroup G of S_n. □

Suppose that G, H are subgroups of the symmetric group S_n and $H \supset G$. Denote by $\pi_{GH} \colon SP^n_G \to SP^n_H$ the natural transformation defined by the formula

$$\pi_{GH}X[x_0, \ldots, x_{n-1}]_G = [x_0, \ldots, x_{n-1}]_H, \ [x_0, \ldots, x_{n-1}]_G \in SP^n_G X.$$

Proposition 3.5.7. *The natural transformation $\pi_{GH} \colon SP^n_G \to SP^n_H$ is also a natural transformation of the extensions of the functors SP^n_G, SP^n_H onto the category $\mathbf{Comp}_{\mathbb{H}}$.*

Proof. In the notation from the proof of Theorem 3.5.6 we have

$$\begin{aligned}
&\exp \pi_{GH} \circ d_G X[A_0, \ldots, A_{n-1}]_G \\
&= \{\pi_{GH}[a_0, \ldots, a_{n-1}]_G \mid a_i \in A_i, \ i \in n\} \\
&= \{[a_0, \ldots, a_{n-1}]_H \mid a_i \in A_i, \ i \in n\} \\
&= d_H X \circ \pi_{GH} \exp X[A_0, \ldots, A_{n-1}]_G.
\end{aligned}$$

Apply Proposition 1.2.10. □

We can prove similar results for the probability measure monad $\mathbb{P}$.

Theorem 3.5.8. *A normal functor F of finite degree $n \geq 1$ can be extended onto the category $\mathbf{Comp}_{\mathbb{P}}$ if and only if F is isomorphic to SP^n_G for some subgroup $G \subset S_n$. There is a unique extension of SP^n_G onto $\mathbf{Comp}_{\mathbb{P}}$.*

Proof. Sufficiency. Define the map $\xi X : SP^n_G(PX) \to P(SP^n_G X)$ by the formula

$$\xi_X\big([m_0, \dots, m_{n-1}]_G\big) = P\pi_G X\Big(\frac{1}{|G|}\sum_{\sigma\in G} m_{\sigma(0)} \otimes \cdots \otimes m_{\sigma(n-1)}\Big),$$

where $m_0, \dots, m_{n-1} \in PX$. Obviously, the map ξX is continuous and $\xi = (\xi X)$ is a natural transformation. Show that this transformation satisfies the conditions of Theorem 1.2.8.

If $[x_0, \dots, x_{n-1}]_G \in SP^n_G X$, then

$$\begin{aligned}\xi X \circ SP^n_G \eta X\big([x_0, \dots, x_{n-1}]_G\big) &= \xi X\big([\delta_{x_0}, \dots, \delta_{x_{n-1}}]_G\big) = \\ &= P\pi_G X\Big(\frac{1}{|G|}\sum_{\sigma\in G} \delta_{x_{\sigma(0)}} \otimes \cdots \otimes \delta_{x_{\sigma n-1}}\Big) = \\ &= P\pi_G X\Big(\frac{1}{|G|}\sum_{\sigma\in G} \delta_{x_{\sigma(0)}\otimes\cdots\otimes x_{\sigma n-1}}\Big) = \delta_{[x_0,\dots,x_{n-1}]_G},\end{aligned}$$

i.e., $\xi \circ SP^n_G = \eta SP^n_G$.

It is sufficient to prove the equality

$$\begin{aligned}&\xi X \circ SP^n_G \psi X\big([M_0, \dots, M_{n-1}]_G\big) \\ &\qquad = \psi SP^n_G X \circ P\xi X \circ \xi PX\big([M_0, \dots, M_{n-1}]_G\big)\end{aligned}$$

only for elements $M_0, \dots, M_{n-1}$ with finite supports (with respect to the functor P^2). Suppose that

$$\mathrm{supp}_{P^2}(M_0) \cup \cdots \cup \mathrm{supp}_{P^2}(M_{n-1}) = \{x_i \mid 0 \le i \le N\}$$

and

$$M_i = \sum_{j(i)=0}^{s} \alpha_{j(i)} \delta(m_{j(i)}),$$

where

$$m_{j(i)} = \sum_{k(j(i))=0}^{N} \beta_{k(j(i))} \delta(x_{k(j(i))}).$$

Then

$$\psi X(M_i) = \sum_{j(i)=0}^{s} \sum_{k(j(i))=0}^{N} \alpha_{j(i)} \beta_{k(j(i))} \delta(x_{k(j(i))})$$

and

$$\psi X(M_{\sigma(0)}) \otimes \cdots \otimes \psi X(M_{\sigma(n-1)}) =$$

$$= \sum_{\substack{j(\sigma(0))=0 \\ \cdots \\ j(\sigma(n-1))=0}}^{s} \sum_{\substack{k(j(\sigma(0)))=0 \\ \cdots \\ k(j(\sigma(n-1)))=0}}^{N} \alpha_{(j(\sigma(0))} \cdots \alpha_{(j(\sigma(n-1))} \times$$

$$\times \beta_{k(j(\sigma(0)))} \cdots \beta_{k(j(\sigma(n-1)))} \delta\big(x_{k(j(\sigma(0)))}, \ldots, x_{k(j(\sigma(n-1)))}\big).$$

This implies that

$$\xi X \circ SP^n_G \psi X\big([M_0, \ldots, M_{n-1}]_G\big) =$$

$$= P\pi_G X\Big(\frac{1}{|G|}\sum_{\sigma\in G}\big(\psi X(M_{\sigma(0)} \otimes \cdots \otimes \psi X(M_{\sigma(n-1)})\big)\Big) =$$

$$= \sum_{\substack{j(\sigma(0))=0 \\ \cdots \\ j(\sigma(n-1))=0}}^{s} \sum_{\substack{k(j(\sigma(0)))=0 \\ \cdots \\ k(j(\sigma(n-1)))=0}}^{N} \alpha_{(j(\sigma(0))} \cdots \alpha_{(j(\sigma(n-1))} \times$$

$$\times \beta_{k(j(\sigma(0)))} \cdots \beta_{k(j(\sigma(n-1)))} \delta\big([x_{k(j(\sigma(0)))}, \ldots, x_{k(j(\sigma(n-1)))}]_G\big).$$

On the other hand,

$$\xi PX\big([M_0, \ldots, M_{n-1}]_G\big) = P\pi_G X\Big(\frac{1}{|G|}\sum_{\sigma\in G} M_{\sigma(0)} \otimes \cdots \otimes M_{\sigma(n-1)}\Big) =$$

$$= \sum_{\substack{j(\sigma(0))=0 \\ \cdots \\ j(\sigma(n-1))=0}}^{s} \alpha_{(j(0)} \cdots \alpha_{(j(n-1)} \delta\big([m_{j(0)}, \ldots, m_{j(n-1)}]_G\big)$$

and, since

$$\xi X\big([m_{j(0)}, \ldots, m_{j(n-1)}]_G\big) =$$

$$= \sum_{\substack{k(j(\sigma(0)))=0 \\ \cdots \\ k(j(\sigma(n-1)))=0}}^{N} \beta_{k(j(\sigma(0)))} \cdots \beta_{k(j(\sigma(n-1)))}$$

$$\times \delta\big([x_{k(j(\sigma(0)))}, \ldots, x_{k(j(\sigma(n-1)))}]_G\big),$$

we immediately obtain the required equality.

Show that the natural transformation $\xi\colon SP^n_G(PX) \to P(SP^n_G X)$ is completely determined by the property $\xi \circ SP^n_G\eta = \eta SP^n_G$. Indeed, let $\xi'\colon SP^n_G P \to PSP^n_G$ be a natural transformation such that $\xi' \circ SP^n_G\eta = \eta SP^n_G$. For every $m \geq 1$ denote by $g_m\colon n \to P(n \times m)$ the following map:

$$g_m(i) = \frac{1}{m}\sum_{j\in m} \delta_{(i,j)}.$$

Let $a \in SP^n_G(n)$ and $\deg(a) = n$. Denoting by $\mathrm{pr}_1\colon n \times m \to n$ the projection, we obtain

$$SP^n_G P\mathrm{pr}_1 \circ SP^n_G g_m(a) = SP^n_G\eta n(a).$$

Therefore,

$$\xi' n \circ SP^n_G P\mathrm{pr}_1 \circ SP^n_G g_m(a) = \eta SP^n_G n(a),$$

and by naturality of ξ' we have

$$PSP^n_G\mathrm{pr}_1 \circ \xi'(n \times m) \circ SP^n_G g_m(a) = \eta SP^n_G n(a).$$

Since the element $\xi'(n \times m) \circ SP^n_G g_m(a)$ is invariant with respect to endomorphisms of $n \times m$ preserving the first coordinate, we obtain

$$\xi'(n \times m) \circ SP^n_G g_m(a) = \frac{1}{n^m}\sum_{f\in\mathfrak{A}} \delta(SP^n_G f(a)),$$

where

$$\mathfrak{A} = \{f\colon n \to n \times m \mid \mathrm{pr}_1 \circ f = \mathrm{id}\}.$$

It is easy to verify that the right part of the previous equation equals to $\xi(n \times m) \circ SP^n_G g_m(a)$.

If $c \in SP^n_G PX$ is represented as $[m_0, \dots, m_{n-1}]_G$, where every m_i is a measure with finite support and $m_i = \sum_{j=0}^{k_i} \alpha_{ij}\delta_{x_{ij}}$, $\alpha_{ij} \in \mathbb{Q}$, then for some $m \geq 1$ there exists a map $f\colon n \times m \to X$ such that $c = SP^n_G Pf(SP^n_G g_m(a))$. Hence, $\xi' X(c) = \xi X(c)$. Since the set of all such points is dense in $SP^n_G X$, we see that $\xi' X = \xi X$. Finally, $\xi' = \xi$.

The necesity can be obtained mutatis mutandis from the proof of Theorem 3.5.6. □

The proof of the following fact is similar to that of Proposition 3.5.7.

Proposition 3.5.9. *The natural transformation $\pi_{GH}: SP^n_G \to SP^n_H$ is also a natural transformation of the extensions of the functors SP^n_G, SP^n_H onto the category $\mathbf{Comp}_{\mathbb{P}}$.*

3.5.1. Extension of hyperspace functor onto the Kleisli category of the hyperspace monad

For every $\mathcal{A} \in \exp^2 X$ let

$$tX(\mathcal{A}) = \{B \in \exp X \mid B \in uX(\mathcal{A}),\ B \cap A \neq \varnothing \text{ for every } A \in \mathcal{A}\}.$$

Lemma 3.5.10. *$tX: \exp^2 X \to \exp^2 X$ is a continuous map.*

Proof. First, check that $tX(\mathcal{A}) \in \exp^2 X$. Indeed, given $C \in \exp X \setminus tX(\mathcal{A})$ we have either $C \not\subset uX(\mathcal{A})$ or $C \cap A = \varnothing$ for some $A \in \mathcal{A}$. In the former case $C \in \langle X, X \setminus uX(\mathcal{A})\rangle \subset \exp X \setminus tX(\mathcal{A})$, in the latter one $C \in \langle X \setminus A\rangle \subset \exp X \setminus tX(\mathcal{A})$.

Prove that the map tX is continuous. Suppose that

$$tX(\mathcal{A}) \in \langle\langle W_{11}, \dots, W_{1n_1}\rangle, \dots, \langle W_{k1}, \dots, W_{kn_k}\rangle\rangle.$$

A subfamily $\mathcal{V}$ of $\mathcal{W} = \{W_{ij} \mid 1 \leq i \leq k,\ 1 \leq j \leq n_i\}$ is called *distinguished* if there is $A \in \mathcal{A}$ such that $\mathcal{V} = \{W \in \mathcal{W} \mid A \cap W \neq \varnothing\}$. Let $\{\mathcal{V}_1, \dots, \mathcal{V}_s\}$ be the set of all distinguished families of $\mathcal{A}$. We will use the following denotation: for any finite family $\mathcal{U} = \{U_1, \dots, U_p\}$ of open subsets in X let $\langle\mathcal{U}\rangle = \langle U_1, \dots, U_p\rangle$. Then $\mathcal{A} \in \langle\langle\mathcal{V}_1\rangle, \dots, \langle\mathcal{V}_s\rangle\rangle$ and

$$tX(\langle\langle\mathcal{V}_1\rangle, \dots, \langle\mathcal{V}_s\rangle\rangle) \subset \langle\langle W_{11}, \dots, W_{1n_1}\rangle, \dots, \langle W_{k1}, \dots, W_{kn_k}\rangle\rangle.$$

□

Lemma 3.5.11. *$t = (tX)$ is a natural transformation of the functor $\exp^2$ into itself.*

Proof. Straightforward. □

Let $t' = s \exp \circ u$.
For every $\mathcal{A} \in \exp^2 X$ let

$$t''X(\mathcal{A}) = \{B \in tX(\mathcal{A}) \mid A \subset B \text{ for some } A \in \mathcal{A}\}.$$

Lemma 3.5.12. *$t''X: \exp^2 X \to \exp^2 X$ is a continuous map.*

Proof. First, show that $t''X(\mathcal{A}) \in \exp^2 X$, i. e. $t''(\mathcal{A})$ is a closed subset of $\exp X$. If $C \in tX(\mathcal{A}) \setminus t''X(\mathcal{A})$, then for every $A \in \mathcal{A}$ there exists an open in X subset V_A such that $C \subset V_A$ and $A \in \langle X, X \setminus V_A \rangle$. The open cover $\{\langle X, X \setminus \bar{V}_A \rangle \mid A \in tX(\mathcal{A})\}$ of the set $tX(\mathcal{A})$ contains a finite subcover $\{\langle X, X \setminus \bar{V}_{A_i} \rangle \mid 1 \leq i \leq l\}$. Then

$$C \in \langle V_{A_1} \rangle \cap \cdots \cap \langle V_{A_1} \rangle \subset \exp^2 X \setminus t''X(\mathcal{A}).$$

Thus, the set $\exp^2 X \setminus t''X(\mathcal{A})$ is open in $\exp^2 X$.

We leave verifying continuity of $t''X$ to the reader (actually, we will not need this fact in the sequel). □

Lemma 3.5.13. *$t'' = (t''X)$ is a natural transformation of the functor $\exp^2$ into itself.*

Proof. Straightforward. □

Lemma 3.5.14. *Suppose that $\xi \colon \exp^2 X \to \exp^2 X$ is a natural transformation such that $\xi \circ \exp s = s \exp$. Then $\xi \in \{t, t', t''\}$.*

Proof. Given a finite ordinal n let

$$\mathcal{A}_n = \{\{(i,j) \mid j \in n\} \mid i \in n\} \in \exp^2(n \times n).$$

Then

$$\begin{aligned}\exp^2 \mathrm{pr}_1 \circ \xi(n \times n)(\mathcal{A}_n) =& \xi n \circ \exp^2 \mathrm{pr}_1(\mathcal{A}_n)\\ &= \xi n \circ \exp sn(n) = s \exp n(n) = \{n\}.\end{aligned}$$

Consider the following three cases.

Case 1. For every n we have

$$\xi(n \times n)(\mathcal{A}_n) = \{n \times n\} = s\exp(n \times n) \circ u(n \times n)(\mathcal{A}_n).$$

Note that for every $\mathcal{B} \in (\exp^2)_\omega X$ there exists a finite ordinal n and a map $f \colon n \times n \to X$ such that $\exp^2 f(\mathcal{A}_n) = \mathcal{B}$. Then we obtain

$$\begin{aligned}\xi X(\mathcal{B}) =& \xi X \circ \exp^2 f(\mathcal{A}_n) = \exp^2 f(\{n \times n\}) =\\ &\{uX(\mathcal{B})\} = s\exp X \circ uX(\mathcal{B}).\end{aligned}$$

Since $(\exp^2)_\omega X$ is dense in $\exp^2 X$, we see that $\xi X = s \exp X \circ uX$, i. e. $\xi = t'$.

Case 2. There exists $n > 1$ such that the set $\xi(n \times n)(\mathcal{A}_n)$ contains an element B such that the following holds: for every $i \in n$ there exists $j \in n$ such that $(i, j) \notin B$.

Denote by $\mathcal{H}_n$ the set of all bijections $h\colon n \times n \to n \times n$ such that $\mathrm{pr}_1 \circ h = g \circ \mathrm{pr}_1$, for some $g\colon n \to n$. Since $\exp^2 h(\mathcal{A}_n) = \mathcal{A}_n$ for every $h \in \mathcal{H}_n$, by naturality of ξ, we see that $\exp^2 h \circ \xi(n \times n)(\mathcal{A}_n) = \xi(n \times n)(\mathcal{A}_n)$, $h \in \mathcal{H}_n$. This implies that the family $\xi(n \times n)(\mathcal{A}_n)$ contains an element B' such that $(i, n-1) \notin B'$ for every $i \in n$.

Consider a finite ordinal $m \geq n$ and denote by $r\colon m \to n$ the retraction such that $r(k) = n - 1$ for every $k \geq n$. By surjectivity of the map $\exp^2 r$, there exists $C_m \in \xi(m \times m)(\mathcal{A}_m)$ such that $(i, k) \notin C_m$, whenever $k \geq n$, $i, k \in m$.

Fix an arbitrary finite ordinal $n' \neq 0$ and $D \subset n' \times n'$ such that $\mathrm{pr}_1(D) = n'$. It is not difficult to find a map $g\colon m \to n'$, for a sufficiently large m, such that

$$\exp^2(g \times g)(\mathcal{A}_m) = \mathcal{A}_{n'}, \quad (g \times g)(C_m) = D.$$

This implies

$$\xi(n' \times n')(\mathcal{A}_{n'}) = \{D' \subset n' \times n' \mid \mathrm{pr}_1(D') = n'\} = t(n' \times n')(\mathcal{A}_{n'}).$$

Arguing as in Case 1 we conclude that $\xi = t$.

Case 3. $\xi \neq t'$ and for every $n \geq 1$, every $B \in \xi(n \times n)(\mathcal{A}_n)$ there exists $i \in n$ such that $\{(i, j) \mid j \in n\} \subset B$.

Since $\xi \neq t'$, for some $n \geq 2$ there exists $B \in \xi(n \times n)(\mathcal{A}_n)$, $B \neq n \times n$. It is easy to construct a map $f\colon n \times n \to 2 \times 2$ such that $\exp^2 f(\mathcal{A}_n) = \mathcal{A}_2$ and

$$\exp^2 f(B) = \{(0,0), (1,0), (1,1))\} \in \xi(2 \times 2)(\mathcal{A}_2).$$

For every $m \geq 2$ let $g\colon m \to 2$ be a map such that $g^{-1}(1) = \{m - 1\}$. Then

$$\exp^2(g \times g) \circ \xi(m \times m)(\mathcal{A}_m) = \xi(2 \times 2)(\mathcal{A}_2),$$

i. e. the family $\xi(m \times m)(\mathcal{A}_m)$ contains the subset $D_m = \{(i, 0) \mid i \in m\} \cup \{(0, i) \mid i \in m\}$.

Now, for an arbitrary $m' > 1$ and $B' \in t'(m' \times m')$ it is easy to construct a map $g'\colon m \times m \to m' \times m'$, for some m, such that $\exp^2 g'(\mathcal{A}_m) = \mathcal{A}_{m'}$ and $\exp^2 g'(D_m) = B'$, whence $\xi(m' \times m')(\mathcal{A}_{m'}) = t''(m' \times m')(\mathcal{A}_{m'})$ and $\xi = t''$. $\square$

The following result gives the classification of the extension of the hyperspace functor onto the Kleisli category of the hyperspace monad.

Theorem 3.5.15. *There are exactly two extensions of the functor* exp *onto the category* $\mathbf{Comp}_{\mathbb{H}}$. *They correspond to the natural transformations* $t, t'\colon \exp^2 \to \exp^2$.

Proof. Show that the natural transformation t satisfies the conditions of Theorem 1.2.8. Note that $uX \circ tX(\mathcal{A}) = uX(\mathcal{A})$ for every $\mathcal{A} \in \exp^2 X$.

Let $A \in \exp X$, then

$$tX \circ \exp sX(A) = tX(\{\{a\} \mid a \in A\}) = \{A\} = s\exp X(A),$$

i. e. $t \circ \exp s = s\exp$.

Let $\mathfrak{A} \in \exp^3 X$. Show that

$$u\exp X \circ \exp tX \circ t\exp X(\mathfrak{A}) = tX \circ \exp uX(\mathfrak{A}).$$

Let $C \in u\exp X \circ \exp tX \circ t\exp X(\mathfrak{A})$, then there exists an element $\mathcal{B} \in t\exp X(\mathfrak{A})$ such that $C \in tX(\mathcal{B})$. For every $\mathcal{A} \in \mathfrak{A}$ we have $\mathcal{B} \cap \mathcal{A} \neq \varnothing$ and, as can be easily seen,

$$C \subset uX(\mathcal{B}) \subset uX \circ u\exp X(\mathfrak{A}) = uX \circ \exp uX(\mathfrak{A}),$$

whence $C \in tX \circ \exp uX(\mathfrak{A})$. Therefore,

$$u\exp X \circ \exp tX \circ t\exp X(\mathfrak{A}) \subset tX \circ \exp uX(\mathfrak{A}).$$

Conversely, suppose that $B \in tX \circ \exp uX(\mathfrak{A})$. Put

$$\mathcal{B} = \{D \in u\exp X(\mathfrak{A}) \mid D \cap B \neq \varnothing\}.$$

Then, obviously, $\mathcal{B} \in t\exp X(\mathfrak{A})$ and $B \in tX(\mathcal{B})$. Consequently, $B \in u\exp X \circ \exp tX \circ t\exp X(\mathfrak{A})$ and we are done.

We have proved that the natural transformation t denermines an extension of exp on $\mathbf{Comp}_{\mathbb{H}}$. For the natural transformation t' the same fact follows from general reasonings. Finally, note that $u\exp \circ \exp t'' \circ t''\exp \neq t'' \circ \exp u$. This finishes the proof. $\square$

Note that for every ordinal α the functor $\exp^\alpha$ has an extension onto the category $\mathbf{Comp}_{\mathbb{H}}$ — this follows from the general results of Section 1.2.

To establish that there are another extensions of $\exp^\alpha$ onto $\mathbf{Comp}_{\mathbb{H}}$ we prove the following

Proposition 3.5.16. *The natural transformation $t\colon \exp^2 \to \exp^2$ satisfies condition 1) of Theorem 1.2.8 (with $\xi = t$, $F = T = \exp$, $\mu = u$).*

Proof. Let $\mathfrak{A} \in \exp^3 X$, then

$$tX \circ u \exp X(\mathfrak{A}) = \{B \in \exp X \mid B \subset uX \circ \exp X(\mathfrak{A}), B \cap A \neq \varnothing\}$$
$$\text{for every } A \in \mathcal{A} \in \mathfrak{A},$$
$$\exp uX \circ$$
$$t \exp X \circ \exp tX(\mathfrak{A}) = \{uX(\mathcal{B})\mathcal{B} \in \exp^2 X, \mathcal{B} \subset$$
$$u \exp X \circ \exp tX(\mathfrak{A}), \mathcal{B} \cap tX(\mathfrak{A}) \neq \varnothing \text{ for all } \mathcal{A} \in \mathfrak{A}\}$$

If $\mathcal{B} \in \exp^2 X, \mathcal{B} \subset u \exp X \circ \exp tX(\mathfrak{A})$ and $\mathcal{B} \cap tX(\mathfrak{A}) \neq \varnothing$ for every $\mathcal{A} \in \mathfrak{A}$, then, assuming that $uX(\mathcal{B}) \cap A \neq \varnothing$ for some $A_0 \in \mathcal{A}_0 \in \mathfrak{A}$, one can find $\mathcal{B} \cap tX(\mathcal{A}_0) \neq \varnothing$. But then for $C \in tX(\mathcal{A}_0) \cap \mathcal{B}$ we have $C \cap A \neq \varnothing$ which leads to a contradiction. Thus, $\exp uX \circ t \exp tX(\mathfrak{A}) \in tX \circ u \exp X(\mathfrak{A})$.

To prove the inverse inclusion, it is sufficient to consider the case of finite X. Let $B \in tX \circ u \exp X(\mathfrak{A})$ and $\mathcal{B} = \{B \cap uX(\mathcal{A}) | \mathcal{A} \in \mathfrak{A}\}$, then $\mathcal{B} \in \exp^2 X, uX(\mathcal{B}) = B, \mathcal{B} \cap tX(\mathcal{A}) \neq \varnothing$ for every $\mathcal{A} \in \mathfrak{A}$. It remains to show that $\mathcal{B} \subset u \exp X \circ \exp tX(\mathfrak{A})$. Choosing $C_0 \in \mathcal{B}$, find $A_0 \in \mathfrak{A}$ such that $C_0 = B \cap uX(\mathcal{A}_0)$. Then $C_0 \in tX(\mathcal{A}_0)$ and $C_0 \in u \exp X \circ \exp tX(\mathfrak{A})$. $\square$

3.5.2. Extension of functors onto the Kleisli category of the inclusion hyperspace monad and some of its submonads

Theorem 3.5.17. *Let F be a weakly normal functor of finite degree $n > 1$. Then F does not extend onto the Kleisli categories of the inclusion hyperspace monad $\mathbb{G}$, the full linked system monad $\mathbb{N}_2$ and the superextension monad $\mathbb{L}$.*

Proof. Assume the contrary, then by Theorem 1.2.8 and the inclusions $\lambda \subset N_2 \subset G$, we see that there exists a natural transformation $\xi = (\xi X)\colon F\lambda \to GF$ with:

1) $\xi \circ F\eta = \eta F$,
2) $\xi \circ F\mu = \mu F \circ G\xi \circ \xi\lambda\colon F\lambda^2 \to GF$.

Recall that the natural transformations η, μ of the monads $\mathbb{G}, \mathbb{N}_2$ and $\mathbb{L}$ is defined by the formulae:

$$\eta X(x) = \{A \in \exp X \mid A \ni x\}, \quad \mu X(\mathfrak{A}) = \bigcup\{\bigcap \mathcal{A} \mid \mathcal{A} \in \mathfrak{A}\},$$

$x \in X$, $\mathfrak{A} \in GX$ (N_2X or λX), $X \in \mathbf{Comp}$.

Let $a \in Fn$ be a point of degree n. Since $\deg F = n$, we have for a retraction $r\colon n+1 \to n$ with $r(n) = n-1$ such that the preimage $(Fr)^{-1}(a)$ of a consists from two points, namely, a and a copy of a in $F\{0, \dots, n-2, n\}$.

Let K be a compact Hausdorff space and $t_1, t_2, t_3 \in K$. Denote by $t(t_1, t_2, t_3)$ the following maximal linked system

$$\{A \in \exp K \mid |A \cap \{t_1, t_2, t_3\}| \geq 2\},$$

whenever all t_i are distinct, and by $a(t_1, t_2)$ denote a copy of a in $\{0, \dots, n-3, t_1, t_2\}$:

$$a(t_1, t_2) = Ff(a),$$

where $f\colon n \to K \sqcup (n-2)$ is an identical on $n-2$ embedding such that $f(n-2) = t_1$, $f(n-1) = t_2$.

Note that for every map $f\colon X \to Y$, $X, Y \in \mathbf{Comp}$, such that $(n-2) \subset X \cap Y$, $f|(n-2) = \mathrm{id}$, and points $M, N \in FX$ the following equality

$$Ff(a(M,N)) = a(f(M), f(N))$$

holds. Indeed, let $f_1\colon n \to X$ and $f_2\colon n \to Y$ act by the formulae

$$f_1(n-2) = M, \ f_1(n-1) = N,$$
$$f_2(n-2) = f(M), \ f_2(n-1) = f(N), \ f_i|(n-2) = \mathrm{id}.$$

Then

$$\begin{aligned} Ff(a(M,N)) &= Ff \circ Ff_1(a) = F(f \circ f_1) \circ Ff_2^{-1}(a(f(M), f(N))) \\ &= F(f \circ f_1 \circ f_2^{-1})(a(f(M), f(N))) \\ &= F\,\mathrm{id}_{f_2(n)}(a(f(M), f(N))) = a(f(M), f(N)). \end{aligned}$$

Consider a compact discrete space $Z = \{x_1, x_2, x_3, y_1, y_2, y_3\} \sqcup (n-2)$, where all points x_i, y_j are distinct. For the sake of simplicity we shall write $\overline{x}, \overline{y}, a_{ij}$ instead of $t(x_1, x_2, x_3)$, $t(y_1, y_2, y_3)$, $a(x_i, y_j)$, respectively.

Remark that the following equalities

$$\xi Z(a(\overline{x}, \eta Z(y_i))) = t(a_{1i}, a_{2i}, a_{3i}), \quad i = 1, 2, 3, \tag{3.10}$$

hold. Indeed, fixing i, denote by $\mathcal{A}$ the left part of the equality and consider a retraction $g\colon Z \to Z \setminus \{x_2\}$ such that $g(x_2) = x_1$. Then

$$\begin{aligned} GFg(\mathcal{A}) &= \xi Z \circ F\lambda g(a(\overline{x}, \eta Z(y_i))) = \xi Z(a(\lambda g(\overline{x}), \lambda g(\eta Z(y_i)))) = \\ &= \xi Z(a(\eta Z(x_1), \eta Z(y_i))) = \xi Z \circ F\eta Z(a_{1i}) = \eta F Z(a_{1i}). \end{aligned}$$

Since $\mathcal{A}$ is an inclusion hyperspace, the set $(Fg)^{-1}(a_{1i}) = \{a_{1i}, a_{2i}\}$ belongs to $\mathcal{A}$. Moreover, every element of the inclusion hyperspace $\mathcal{A}$ intersects $(Fg)^{-1}(a_{1i})$. Similarly, $\mathcal{A}$ contains the sets $\{a_{1i}, a_{3i}\}$ and $\{a_{2i}, a_{3i}\}$. Now since every element of $\mathcal{A}$ intersects the sets $\{a_{1i}, a_{2i}\}$, $\{a_{1i}, a_{3i}\}$, $\{a_{2i}, a_{3i}\}$, we see that every element of $\mathcal{A}$ contains at least one of these sets.

Consider the following inclusion hyperspace

$$\mathcal{B} = \xi Z(a(\overline{x}, \overline{y})). \tag{3.11}$$

Similarly, one can easy obtain (using the retraction $Z \to Z \setminus \{x_2, y_2\}$, $x_2 \mapsto x_1$, $y_2 \mapsto y_1$) that $\{a_{11}, a_{12}, a_{21}, a_{22}\} \in \mathcal{B}$.

Now show that $\mathcal{B}$ contains the set $\{a_{11}, a_{12}, a_{22}, a_{31}\}$. For this consider the points $\mathcal{M}, \mathcal{N} \in \lambda^2 Z$:

$$\mathcal{M} = t(\overline{x}, \eta Z(x_1), \eta Z(x_2)), \quad \mathcal{N} = t(\eta Z(y_1), \eta Z(y_2), \overline{y}).$$

Since $\mu Z(\mathcal{M}) = \overline{x}$, $\mu Z(\mathcal{N}) = \overline{y}$, we have

$$\xi Z \circ F\mu Z(a(\mathcal{M}, \mathcal{N})) = \xi Z(a(\overline{x}, \overline{y})) = \mathcal{B}.$$

Applying property 2) of ξ, we obtain that

$$\mathcal{B} = \mu F Z \circ G\xi Z \circ \xi \lambda Z(a(\mathcal{M}, \mathcal{N})). \tag{3.12}$$

If in (3.11) replace $\overline{x}, \overline{y}$ by $\mathcal{M}$ and $\mathcal{N}$, respectively, then it is easy to see that the set

$$L = \Big\{a(\overline{x}, \eta Z(y_1)), a(\overline{x}, \eta Z(y_2)), \\ a(\eta Z(x_1), \eta Z(y_1)), a(\eta Z(x_1), \eta Z(y_2))\Big\}$$

belongs to $\xi\lambda Z(a(\mathcal{M},\mathcal{N}))$. Clearly, $\bigcap\xi Z(L)\subset\mathcal{B}$ (see (3.12)). Now, using (3.10) and the equality

$$\xi Z\big(a(\eta Z(x_i),\eta Z(y_j))\big)=\xi Z\circ F\eta Z(a_{ij})=\eta FZ(a_{ij}),$$

we obtain that $\{a_{11},a_{31},a_{12},a_{22}\}\in\mathcal{B}$.

Consider the point $\mathcal{R}=a\big(t(\overline{x},\eta Z(x_2),\eta Z(x_3)),\eta\lambda Z(\overline{y})\big)\in F\lambda^2Z$. Since $\mu Z\circ\eta\lambda Z=\mathrm{id}_{\lambda Z}$, we have

$$\xi Z\circ F\mu Z(\mathcal{R})=\xi Z\big(a(\overline{x},\overline{y})\big)=\mathcal{B}.$$

Therefore, $\mathcal{B}=\mu FZ\circ G\xi Z\circ\xi\lambda Z(\mathcal{R})$. Using the equality

$$\xi\lambda Z(\mathcal{R})=t\big(a(\overline{x},\overline{y}),a\big(\eta Z(x_2),\overline{y}\big),a\big(\eta Z(x_3),\overline{y}\big)\big)$$

(one can easy verify it, making obvious substitutions in (3.10)), we obtain that the set $\{a_{11},a_{12},a_{22},a_{31}\}\in\mathcal{B}$ must belong to at least two of the following inclusion hyperspaces:

$$\xi Z\big(a(\overline{x},\overline{y})\big),\quad \xi Z\big(a\big(\eta Z(x_2),\overline{y}\big)\big),\quad \xi Z\big(a\big(\eta Z(x_3),\overline{y}\big)\big).$$

However, this set is an element of neither the set $\xi Z\big(a\big(\eta Z(x_2),\overline{y}\big)\big)$ nor the set $\xi Z\big(a\big(\eta Z(x_3),\overline{y}\big)\big)$ (because the equalities

$$\xi Z\big(a\big(\eta Z(x_i),\overline{y}\big)\big)=t(a_{i1},a_{i2},a_{i3}),\quad i=1,2,3,$$

hold similarly to (3.10)). This gives the contradiction. □

Theorem 3.5.18. *Let F:* **Comp** $\to$ **Comp** *be a weakly normal functor of finite degree $n>1$. Then F does not exist onto the Kleisli category of the k-linked system monad $\mathbb{N}_k$, $k\geq 3$.*

Sketch of the proof. To the contrary, there exists a natural transformation $\xi=(\xi X)\colon FN_k\to N_kF$ with
1) $\xi\circ F\eta=\eta F$,
2) $\xi\circ F\mu=\mu F\circ N_k\xi\circ\xi N_k$.

For a compact Hausdorff space K and points $t_1,\dots,t_{k+1}\in K$ define the point $a(t_1,t_2)$ as in the previous theorem, and (whenever all t_i are distinct) denote by $t(t_1,\dots,t_{k+1})$ the k-linked system

$$\big\{A\in\exp K\mid |A\cap\{t_1,\dots,t_{k+1}\}|\geq k\big\}.$$

Let

$$Z = \{x_1, \dots, x_{k+1}, y_1, \dots, y_{k+1}\} \sqcup (n-2);$$
$$\overline{x} = t(x_1, \dots, x_{k+1}), \quad \overline{y} = t(y_1, \dots, y_{k+1}), \quad a_{ij} = a(x_i, y_j),$$

where all x_i, y_j are distinct.

Let $\mathcal{A} = \xi Z\big(a(\overline{x}, \overline{y})\big)$. Considering the following k-linked systems in $N_k Z$,

$$\mathcal{M} = t\big((\eta Z(x_1), \dots, \eta Z(x_k), \overline{x}\big), \quad \mathcal{N} = t\big((\eta Z(y_1), \dots, \eta Z(y_k), \overline{y}\big),$$

we obtain that

$$\xi Z \circ F\mu Z\big(a(\mathcal{M}, \mathcal{N})\big) = \xi Z\big(a(\overline{x}, \overline{y})\big) = \mathcal{A}.$$

Hence, by property 2) of ξ we have

$$\mathcal{A} = \mu F Z \circ N_k \xi Z \circ \xi N_k Z\big(a(\mathcal{M}, \mathcal{N})\big).$$

Applying similar to the proof of Theorem 3.5.17 methods to the following sets

$$\mathcal{K} = \Big\{a\big(\eta Z(x_l), \eta Z(y_m)\big) \mid m \le k, l = 2, \dots, k\Big\}$$
$$\cup \Big\{a\big(\overline{x}, \eta Z(y_m)\big) \mid m \le k\Big\},$$
$$\mathcal{L} = \Big\{a\big(\eta Z(x_m), \eta Z(y_l)\big) \mid m \le k, l = 2, \dots, k\Big\}$$
$$\cup \Big\{a\big(\eta Z(x_m), \overline{y}\big) \mid m \le k\Big\},$$

one can prove that the k-linked system $\mathcal{A}$ contains the elements

$$A = \{a_{11}, \dots, a_{k1}\} \cup \{a_{lm} \mid l = 2, \dots, k+1; m = 2, \dots, k\},$$
$$A_1 = \{a_{11}, \dots, a_{1k}\} \cup \{a_{ml} \mid l = 2, \dots, k+1; m = 2, \dots, k\}.$$

Define an authomorphism $h_i \colon Z \to Z$, $i = 2, \dots, k$, by the formulae

$$h_i(x_1) = x_i, \; h_i(x_i) = x_{k+1}, \; h_i(x_{k+1}) = x_1,$$
$$h_i(y_i) = y_{k+1}, \; h_i(y_{k+1}) = y_i,$$

and $h_i(t) = t$ for other $t \in Z$. Put

$$A_i = N_k h_i(A), \; i = 2, \dots, k.$$

Since the k-linked systems $\overline{x}, \overline{y}$ is invariant with respect to the identical on $n-2$ authomorphisms of Z which keep the sets $\{x_1, \dots, x_{k+1}\}$, $\{y_1, \dots, y_{k+1}\}$, we have that $A_i \in \mathcal{A}$. One can verify that $\bigcap_{i=1}^{k} A_i = \varnothing$. But $\mathcal{A}$ is a k-linked system. Contradiction. $\square$

3.5.3. Extensions of contravariant functors onto the Kleisli categories

Every object A of the category $\mathcal{C}$ determines a contravariant functor $\mathrm{Hom}(-, A) : \mathcal{C} \to \mathbf{Set}$:

$$\mathrm{Hom}(X, A) = \mathcal{C}(X, A), \quad \mathrm{Hom}(f, A)(g) = g \circ f.$$

If $\mathcal{C}$ is a subcategory of the category **Top**, the functor $\mathrm{Hom}(-, A)$ admits a natural topologization which will be denoted by $A^{(-)}$:

$$A^X = \prod_{x \in X} A_x, \quad A_x \equiv A$$

(the product is considered in Tychonov topology). We will also denote the subfunctor $A^{\langle - \rangle}$ of the functor $A^{(-)}$:

$$A^{\langle X \rangle} = \overline{\mathcal{C}(X, A)} \subset A^X$$

($\overline{(-)}$ is the closure in A^X).

Theorem 3.5.19. *Let $\mathbb{T}$ be a monad on the category* **Tych** *generated by a normal functor with finite supports and (A, α) a $\mathbb{T}$-algebra. Then the functors $A^{(-)}$ and $A^{\langle - \rangle}$ admit extensions onto* $\mathbf{Tych}_{\mathbb{T}}$.

Proof. For every $X \in |\mathbf{Tych}|$ denote the map $\xi' X : A^X \to A^{TX}$ by the formula $\xi' X(f) = \alpha \circ Tf$, $f \in A^X$. Note that, in spite of the fact that f is not necessarily continuous (i. e. is not a morphism in **Tych**), the map $\xi' X$ is well-defined. Indeed, if $a \in TX$ then, since T is a functor with finite supports, there exist a finite (and hence discrete) space Y, $b \in TY$ and a map $g : Y \to X$ such that $a = Tg(b)$. Put $Tf(a) = T(f \circ g)(b)$ (note that $f \circ g$ is defined on a discrete space and, therefore, is continuous).

Show that $\xi' X$ is continuous. For this, it is sufficient to prove that for every $a \in TX$ so is the composition $\mathrm{ev}_a \circ \xi' X$ (here $\mathrm{ev}_a : A^{TX} \to A$ is the evaluation map, $\mathrm{ev}_a(f) = f(a)$, $f \in A^{TX}$). Let $\xi' X(f_0) = \varphi_0$ and U be a neighborhood of the point $\mathrm{ev}_a(\varphi_0)$. Suppose that $\mathrm{supp}(a) = \{x_1, \dots, x_k\}$. Denote by $h : \{1, \dots, n\} \to \mathrm{supp}(a)$ the map for which $h(i) = x_i$, and let $b \in T\{1, \dots, n\}$ be such that $Th(b) = a$. By Proposition 2.6.6, the map

$$T\{1, \dots, n\} \times C(\{1, \dots, n\}, X), \quad (c, g) \mapsto Tg(c),$$

is continuous. This implies that there exist neighborhoods $V_1, \ldots, V_k$ B A of the points $f_0(x_1), \ldots, f_0(x_k)$ respectively for which the following holds: for every $f \in A^X$ with $f(x_i) \in V_i$, $i = 1, \ldots, k$, we have $T(fh)(b) = Tf(a) \in U$.

We obtain

$$\xi X(f) = \eta A^{TX}(\alpha \circ Tf). \tag{3.13}$$

The equality

$$\begin{aligned} T(A^{\eta X}) \circ \eta A^{TX}(\alpha \circ Tf) &= \eta A^X \circ A^{\eta X}(\alpha \circ Tf) \\ =&\eta A^X(\alpha \circ Tf \circ \eta X) = \eta A^X(\alpha \circ \eta A \circ f) = \eta A^X(f) \end{aligned}$$

implies $T(A^\eta) \circ \xi = \eta A^{(-)}$.

Besides,

$$\begin{aligned} T(A^{\mu X}) \circ \eta A^{TX}(\alpha \circ Tf) &= \eta A^{T^2X} \circ A^{\mu X}(\alpha \circ Tf) \\ =&\eta A^{T^2X}(\alpha \circ Tf \circ \mu X) = \eta A^{T^2X}(\alpha \circ \mu X \circ T^2 f) \\ =&\eta A^{T^2X}(\alpha \circ T(\alpha \circ Tf)) = \xi TX(\alpha \circ Tf) \\ =&\mu A^{T^2X} \circ \eta T A^{T^2X} \circ \xi TX(\alpha \circ Tf) \\ &\mu A^{T^2X} \circ T\xi TX \circ \eta A^{TX}(\alpha \circ Tf) \\ =&\mu A^{T^2X} \circ T\xi TX \circ \xi X(f). \end{aligned}$$

By Theorem 1.2.16, there exists an extension of the functor $A^{(-)}$ onto the category $\mathbf{Tych}_{\mathbb{T}}$.

Replacing in formula 3.13 $f \in C(X, A)$ by $\xi X(f) \in C(TX, TA)$ and passing to closure we obtain that $\xi X(f) \in TA^{\langle TX \rangle}$ for every $f \in A^{\langle X \rangle}$. This means that the functor $A^{\langle - \rangle}$ can be extended onto the category $\mathbf{Tych}_{\mathbb{T}}$. □

Exercises

1. Show that there exists a unique extension of the functor SP^n_G onto the category $\mathbf{Comp}_{\mathbb{H}^c}$.
2. Prove the counterpart of Theorem 3.5.19 for the category $\mathbf{Comp}^\infty$.
3. Is there a (unique) extension of the functor SP^n_G onto the Kleisli category of the monad generated by the functor P^c?

Problems

1. Suppose there exists an extension of a functor F of finite degree onto the Kleisli category of a nonprojective normal monad. Is F isomorphic to SP^n_G for some subgroup $G \subset S_n$?
2. Suppose that a functor F of finite degree n extends to the category $\mathbf{Comp}_{\mathbb{H}^c}$. Is F isomorphic to the functor SP^n_G, for some subgroup G of S_n?

3.6. Eilenberg-Moore categories of some weakly normal monads

The main results of this section are devoted to descriptions of the categories of $\mathbb{T}$-algebras, for some (weakly, almost) normal monads $\mathbb{T}$. Besides, we consider the problem of lifting of functors onto the category of $\mathbb{T}$-algebras.

3.6.1. Category of $\mathbb{H}$-algebras

A *semilattice* is a set endowed with an associative, commutative, idempotent operation. A semilattice operation $\vee$ on a set X induces a partial order relation on X: $x \leq y$ if and only if $x \vee y = y$.

If $X \in |\mathbf{Comp}|$, a semilattice $(X, \vee)$ is called a *compact Hausdorff semilattice* if the map $\vee \colon X \times X \to X$ is continuous. The compact Hausdorff semilattices and their morphisms (the continuous maps that preserve the lattice operation) form the category $\mathbf{CSL}$.

A compact Hausdorff semilattice is said to *have small semilattices* or to be a *Lawson semilattice* if it posesses a base consisisting of subsemilattices. It is proved in Lawson [1969] that a compact Hausdorff semilattice is a Lawson semilattice if and only if it is isomorphic to a subsemilattice of the product of a family of the unit intervals with max as the semilattice structure.

For a finite nonempty subset B of a semilattice X we denote by $\sup B$ the least upper bound of B with respect to the partial order defined above.

Let X be a Lawson semilattice and $A \in \exp X$. Consider the net

$$(\sup B \mid B \text{ is a finite nonempty subset of } A).$$

It is easy to see that this net is convergent and we denote its limit by $\sup A$. It is proved in Lawson [1969] that the continuity of the map $\sup \colon \exp X \to X$ characterizes the Lawson semilattices.

Theorem 3.6.1. *The categories* $\mathbf{Comp}^{\mathbb{H}}$ *and* **CSL** *are isomorphic.*

Proof. Given $(X, \xi) \in \mathbf{Comp}^{\mathbb{H}}$, define the semilattice structure $\vee$ in X by the formula $x \vee y = \xi(\{x, y\})$. On the other hand, note that for every $(X, \vee) \in |\mathbf{CSL}|$ the map $\sup\colon \exp X \to X$ determines the $\mathbb{H}$-algebra structure on X. □

3.6.2. Algebras for the probability measure monad

Denote by **Conv** the category of compact convex subsets of locally convex spaces and affine continuous maps. If $X \in |\mathbf{Conv}|$, one can define the *barycenter map* $b_X : PX \to X$ by the following condition. For every affine function $\varphi \in C(X)$ we have $\varphi(b_X(\mu)) = \mu(\varphi)$, $\mu \in PX$. If $\mu = \sum_{i=1}^k \alpha_i \delta_{x_i}$, then $bX(\mu) = \sum_{i=1}^k \alpha_i x_i$.

Proposition 3.6.2. *For every* $X \in |\mathbf{Conv}|$ *the pair* (X, b_X) *is a* $\mathbb{P}$-*algebra.*

Proof. We have $b_X \circ \eta X(x) = b_X(\delta_x) = x$. Since the set of measures of the form $M = \sum_{i=1}^k \alpha_i \delta_{\mu_i}$, where $\mu_i = \sum_{j=1}^{n_i} \beta_{ij} \delta_{x_{ij}}$, is dense in $P^2 X$, the proof is a consequence of the equality

$$b_X \circ \psi X(M) = \sum_{i=1}^k \alpha_i \sum_{j=1}^{n_i} \beta_{ij} x_{ij} = b_X \circ PbX(M).$$

□

Proposition 3.6.3. *Let* $f\colon X \to Y$ *be a morphism in the category* **Conv**. *Then* $f\colon (X, b_X) \to (Y, b_Y)$ *is a morphism of* $\mathbb{P}$-*algebras.*

Proof. Obvious. □

Definition 3.6.4. A space $X \in |\mathbf{Conv}|$ is called *barycentrically open* if the barycenter map $bX : PX \to X$ is open.

Examples. There exists a compact convex subset of $\mathbb{R}^3$ that is not barycentrically open. Indeed, let X be the convex hull of the set

$$\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \cup \{(1, 0, 1), (1, 0, -1)\}.$$

Indeed, the barycenter map $b_X : PX \to X$ satisfies the following properties:
1) $b_X^{-1}(x, y, 0) = \delta_{(x,y,0)}$, whenever $x^2 + y^2 = 1$ and $x \neq 1$;
2) $b_X^{-1}(1, 0, 0) \neq \delta_{(1,0,0)}$.
Thus, the map $b_X^{-1} : X \to \exp PX$ is not continuous and hence the map b_X is not open.

Proposition 3.6.5. *Let $f\colon X \to Y$ be an affine retraction of convex compact subspaces of a locally convex space. If X is barycentrically open, then so is Y.*

Proof. Since the map f is a morphism in $\mathbf{Comp}^{\mathbb{P}}$, the diagram

$$\begin{tikzcd}
 & b_X^{-1}(Y) \arrow[dl, "P(f)|b_X^{-1}(Y)"'] \arrow[dr, "b_X|b_X^{-1}(Y)"] & \\
P(Y) \arrow[rr, "b_Y"] & & Y
\end{tikzcd}$$

is commutative. The map $Pf|b_X^{-1}(Y)$ is an onto map. Then the map b_Y is open, being a left divisor of an open map $b_X|b_X^{-1}(Y)$. □

Proposition 3.6.6. *Let $f\colon X \to Y$ be an open affine map of convex compact subspaces of a locally convex space. If X is barycentrically open, then so is Y.*

Proof. Being a left divisor of the open map $f \circ b_X$, the barycentric map b_Y is open. □

Further, K is a convex compact subspace of a locally convex space E.

Lemma 3.6.7. *For every point $x \in K$ the sets of measures with finite supports is dense in $b_K^{-1}(x)$.*

Proof. Let $\mu \in b_k^{-1}(x)$ and $U = \{\mu' \in PX \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, 1 \le i \le k\}$ be a neighborhood of μ , where $\varphi_1, \ldots, \varphi_k \in C(K), \varepsilon > 0$.

For every $a \in K$ there is a closed convex neighborhood V_a of 0 such that

$$|\varphi_i(y) - \varphi_i(a)| < \frac{\epsilon}{2}$$

for every $y \in W_a = K \cap (a + V_a)$. Since K is compact, there exists a finite set $\{a_1, \ldots, a_r\} \subset K$ such that $\operatorname{Int} W_{a_1} \cup \cdots \cup \operatorname{Int} W_{a_r} = K$. There exists a partition of unity $\{g_i | 1 \le i \le r\}$ on K with respect to tne cover $\{\operatorname{Int} W_{A_i} | i \le i \le r\}$. Let $\alpha_i = \mu(g_i)$. If $\alpha_i \ne 0$, let $\mu_i = \alpha_i^{-1} g_i \mu$, and if $\alpha_i = 0$, let $\mu_i = \delta_{a_i}$. Then $\mu_i \in PK$ and $\operatorname{supp}(\mu_i) \subset W_{a_i}$.

We have

$$\mu = \sum_{i=1}^{r} \alpha_i \mu_i \tag{3.14}$$

and

$$\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \mu(g_i) = \mu(\sum_{i=1}^{r} g_i) = 1.$$

Obviously, $x_i = b_K(\mu_i) \in W_{a_i}$. Let $\nu = \sum_{i=1}^{r} \alpha_i \delta_{x_i}$, then by (3.14) $b_K(\nu) = b_K(\mu) = x$.

Since $\operatorname{supp}(\mu_j) \subset W_{a_j}$ and $|\varphi_i(y) - \varphi_i(a)| < \frac{\varepsilon}{2}$ for every $y \in W_{a_j}$ and $1 \le i \le k$, we see that

$$|\mu_j(\varphi_i) - \varphi_i(a_j)| < \frac{\varepsilon}{2}, \; 1 \le i \le k.$$

On the other hand, since $x_i \in W_{a_j}$, we have also $|\delta_{x_j}(\varphi_i) - \varphi_i(a_j)| < \frac{\varepsilon}{2}$ for $1 \le i \le k$, whence $|\mu_j(\varphi_j) - \delta_{x_j}(\varphi_j)| < \varepsilon$ for every i, j.

Finally, note that $\alpha_j \ge 0$ and $\sum_{i=1}^{r} \alpha_j = 1$, therefore (3.14) implies that $\nu \in U$. □

For $n \in \mathbb{N}$ let $c_n(x) = \{\{y_1, \dots, y_n\} \in \exp_n K \mid x \in \operatorname{conv}\{y_1, \dots, y_n\}\}$.

Lemma 3.6.8. *The set $c_n(x)$ is closed in $\exp_n K$.*

Proof. Indeed, let $a = \{y_1, \dots, y_n\} \in \exp_n K \setminus c_n(x)$. This means that there exists a symmetric convex neighborhood V of the origin in E such that $(x + V) \cap \operatorname{conv}\{y_1, \dots, y_n\} = \varnothing$. Without loss of generality we may assume that $(y_i + V) \cap (y_j + V) = \varnothing$, whenever $i \neq j$. Then

$$U = \langle y_1 + V, \dots, y_n + V\rangle \cap \exp_n K$$

is a neighborhood of a such that $U \cap c_n(x) = \varnothing$. □

Thus, the map $c_n \colon K \to \exp(\exp_n K)$ is well-defined.

Theorem 3.6.9. *The barycenter map b_K is open if and only if for every $n \in \mathbb{N}$ the map c_n is continuous.*

Proof. Sufficiency. Suppose that for some $\mu \in PK$ there exists a convex neighborhood U such that the set $b_K(U)$ is not a neighborhood of the point $x = b_K(\mu)$. This means that there exists a converging to x net $(x_\alpha)_{\alpha \in A}$ in the complement of $b_K(U)$.

By Lemma 3.6.7, the set $b_K^{-1}(x) \cap U$ contains a measure $\nu = m_1\delta_{y_1} + \dots + m_n\delta_{y_n}$, $m_i > 0$. Then $x = b_K(\nu) = m_1 y_1 + \dots + m_n y_n$. Then for $a = \{y_1, \dots, y_n\}$ we have $a \in c_n(x)$. Since the map c_n is continuous,

there exists elements $a_\alpha \in c_n(x_\alpha)$ such that the net $(a_\alpha)_{\alpha\in\mathcal{A}}$ converges to a. Without loss of generality we may assume that every element a_α consists of exactly n points: $a_\alpha = \{y_1^\alpha, \dots, y_1^\alpha\}$, and the net $(y_i^\alpha)_{\alpha\in\mathcal{A}}$ converges to y_i, $i = 1, \dots, n$.

Let $C = \mathrm{conv}\{y_1, \dots, y_n\}$ and $C_\alpha = \mathrm{conv}\{y_1^\alpha, \dots, y_1^\alpha\}$. Since all m_i are positive, x_i is an inner point of the polyhedron C. Thus, without loss of generality, we may assume that every x_i is an inner point of the polyhedron C_α.

Now let $z_\alpha = m_1 y_1^\alpha + \dots + m_n y_n^\alpha$. For a positive γ, denote by $g_\gamma^\alpha \colon E \to E$ the homothety with coefficient γ and center z_α. Since x_α is an inner point of the polyhedron C_α, there exists $\gamma = \gamma(\alpha) > 0$ such that

$$g_\gamma^\alpha(C_\alpha) + (x_\alpha - z_\alpha) \subset C_\alpha. \tag{3.15}$$

Denote by γ_α the supremum of all γ for whigh 3.15 holds. Since the net $(z_\alpha)_{\alpha\in\mathcal{A}}$ converges to x, the net $(\gamma_\alpha)_{\alpha\in\mathcal{A}}$ converges to 1.

Now let $u_i^\alpha = g_{\gamma_\alpha}^\alpha(y_i^\alpha) + x_\alpha - z_\alpha$ and $\nu_\alpha = m_1\delta_{u_1}\alpha + \dots + m_n\delta_{u_n^\alpha}$. It is easy to see that $b_K(\nu_\alpha) = x_\alpha$. Besides, $\nu_\alpha \to \nu$, because $u_i^\alpha \to y_i$. Consequently, $\nu_\alpha \in U$, whenever $\alpha \geq \alpha_0$, for some $\alpha_0 \in \mathcal{A}$. But this contradicts to the fact that $x_\alpha \notin b_K(U)$ and $x_\alpha = b_K(\nu_\alpha)$. Sufficiency is proved.

Necessity. Suppose that a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ converges to a point x and $a = \{y_1, \dots, y_n\} \in c_n(x)$. First suppose that x is an inner point of a polyhedron $C = \mathrm{conv}\{y_1, \dots, y_n\}$. Then there exists positive numbers $m_1, \dots, m_n$ such that $m_1 + \dots + m_n = 1$ and $x = m_1 y_1 + \dots + m_n y_n$. Let

$$\mu = m_1\delta_{y_1} + \dots + m_n\delta_{y_n}.$$

Obviosly, $b_K(\mu) = x$. Since the map b_K is open, there exists a converging to μ net $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ such that $b(\mu_\alpha) = x_\alpha$. Consider disijoint convex neighborhoods $V_1, \dots, V_n$ of the points $y_1, \dots, y_n$ respectively. Let $\mu_\alpha^i = \mu_\alpha|V_i$, $i = 1, 2, \dots, n$, and $\mu_\alpha^{n+1} = \mu_\alpha|(K \setminus (V_1 \cap \dots \cap V_n))$. Then $\mu_\alpha = \mu_\alpha^1 + \dots + \mu_\alpha^n + \mu_\alpha^{n+1}$, where all μ_α^i are nonnegative, moreover μ_α^i are positive for $i = 1, \dots, n$. Besides, $\mu_\alpha^{n+1} \to 0$ and $\mu_\alpha^i \to \mu|V_i$ for $i = 1, \dots, n$.

Let $m_\alpha^i = \|\mu_\alpha^i\|$ and denote by y_α^i the barycenter of μ_α^i. Then

$$x_\alpha = m_\alpha^1 y_\alpha^1 + \dots + m_\alpha^n y_\alpha^n + m_\alpha^{n+1} y_\alpha^{n+1} \tag{3.16}$$

Note that in the case $m_\alpha^{n+1} = 0$ the point y_α^{n+1} in (3.16) is not uniquely determined (in fact, if can be choosed arbitrarily). Now, for $i = 1, \ldots, n$, let

$$z_\alpha^1 = \begin{cases} y_\alpha^i, & \text{if } m_\alpha^{n+1} = 0, \\ g^\alpha_{1-m_\alpha^{n+1}}(y_\alpha^i), & \text{if } m_\alpha^{n+1} > 0, \end{cases}$$

where g_i^α denote the homothety with center y_α^{n+1} and coefficient t. The definition of z_α^i immediately implies

$$x_\alpha = \tilde{m}_\alpha^1 z_\alpha^1 + \cdots + \tilde{m}_\alpha^n z_\alpha^n \tag{3.17}$$

where $\tilde{m}_\alpha^i = \dfrac{m_\alpha^i}{1 - m_\alpha^{n+1}}$.

Since $y_\alpha^i = b_K(\mu_\alpha^i), \mu_\alpha^i \to \mu|V_i = m_i \delta_{y_i}$, we have $y_\alpha^i \to y_i$. Then also $z_\alpha^i \to y_i$, because $m_\alpha^{n+1} = 0$. Putting $a_\alpha = \{z_\alpha^1, \ldots, z_\alpha^n\}$, we obtain $a_\alpha \to a$ and $a_\alpha \in c_n(x_\alpha)$, by (3.17). Thus, the assertion is proved in the case of inner point of the polyhedron C.

Now suppose that x is not inner. Then x is an inner point of some face C' of C. Renumerating, if necessary, the point $y_1, \ldots, y_n$, we may suppose that $C' = \operatorname{conv}\{y_1, \ldots, y_k\}$ for some $k < n$. Arguing as above, we conclude that there exist points $a'_\alpha = \{y_1^\alpha, \ldots, y_k^\alpha\} \in c_k(x_\alpha)$ such that the net $(a'_\alpha)_{\alpha \in \mathcal{A}}$ converges to $a' = \{y_1, \ldots, y_k\}$. Then the net $(a_\alpha = a' \cap \{y_{k+1}, \ldots, y_n\})_{\alpha \in \mathcal{A}}$ converges to a. □

Now we are able to prove that the probability measure monad is open.

Theorem 3.6.10. *For every $X \in |\mathbf{Comp}|$ the space PX is barycentrically open.*

Proof. If X is finite, then PX is a $(|X| - 1)$-dimensional simplex and it is easy to verify that the conditions of Theorem 3.6.9 hold. Thus, the map $\mu X = b_{PX}$ is open.

Now suppose that X is zero-dimensional. Represent X as the limit of an inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ consisting of finite spaces and epimorphisms. Then $\psi X = \varprojlim(\psi X_\alpha)$. By Proposition 2.10.9, in order to prove that the map ψX is open it is sufficient to prove that the

diagram

$$
\begin{tikzcd}
P^2X_\beta \arrow[r, "\psi X_\beta"] \arrow[d, "P^2p_{\beta\alpha}"'] & PX_\beta \arrow[d, "Pp_{\beta\alpha}"] \\
P^2X_\alpha \arrow[r, "\psi X_\alpha"] & PX_\alpha.
\end{tikzcd}
$$

is bicommutative, $\alpha \leq \beta$. Let $\mu_\beta \in PX_\beta$, $\nu_\alpha \in P^2X_\alpha$ be such that $\psi X_\alpha(\nu_\alpha) = Pp_{\beta\alpha}(\mu_\beta)$. By Lemma 3.6.7, we may consider only the case $\nu_\alpha \in P^2_\omega X_\alpha$. If

$$\nu_\alpha = \sum_{i=1}^{r} a_i\delta\left(\sum_{j=1}^{s(i)} a_{ij}\delta_{x_{ij}}\right),$$

then

$$\psi X_\alpha(\nu_\alpha) = \sum_{i=1}^{r}\sum_{j=1}^{s(i)} a_i a_{ij}\delta_{x_{ij}}$$

and therefore

$$\mu_\beta = \sum_{i=1}^{r}\sum_{j=1}^{s(i)}\sum_{k=1}^{t(i,j)} b_{ijk}\delta_{y_{ijk}},$$

where $p_{\beta\alpha}(y_{ijk}) = x_{ij}$ and $\sum_{k=1}^{t(i,j)} b_{ijk} = a_i a_{ij}$. Let

$$\nu_\beta = \sum_{i=1}^{r} a_i\delta\left(\sum_{j=1}^{s(i)}\sum_{k=1}^{t(i,j)} \frac{b_{ijk}}{a_i}\delta_{y_{ijk}}\right).$$

Then $P^2p_{\beta\alpha}(\nu_\beta) = \nu_\alpha$ and $\psi X_\beta(\nu_\beta) = \nu_\alpha$.

Given an arbitrary $X \in |\mathbf{Comp}|$, we can find a Milutin epimorphism $f\colon Y \to X$, where $Y \in |\mathbf{Comp}_0|$ (see Proposition 1.1.9). Then, by Proposition 2.1.16 PX is an affine retract of PY and by Proposition 3.6.5 PX is barycentrically open.

□

3.6.3. Algebras of the superextension monad

Here we characterize the category of algebras of the superextension monad $\mathbb{L}$.

Let $\mathcal{S}$ be a closed subbase of a compact Hausdorff space X. Recall that $\mathcal{S}$ is *binary* if the intersection of every its linked subsystem is non-empty. A subbase $\mathcal{S}$ is said to be *T_2-subbase* if for every $x_1, x_2 \in X$ there exists $S_1, S_2 \in \mathcal{S}$ such that $S_1 \cup S_2 = X$, $x_1 \notin S_1$, $x_2 \notin S_2$. It is *almost normal* if for every $S \in \mathcal{S}$ and neighborhood OS of S there exists $S_1 \in \mathcal{S}$ such that $S \subset \operatorname{Int} S_1 \subset S_1 \subset OS$.

For every X the space λX is often considered with the following natural subbase

$$\mathcal{L}(X) = \{A^+ \mid A \in \exp X\}, \text{ where } A^+ = \{\mathcal{M} \in \lambda X \mid A \in \mathcal{M}\}.$$

It is not difficult to show that this subbase is T_2, binary, and almost normal (see Zarichnyi [1987b]).

For every $A \subset X$ set

$$I_{\mathcal{S}}(A) = \bigcap \{S \in \mathcal{S} \mid A \subset S\}.$$

Denote by $q_{\mathcal{S}}(X)$ the closure of set

$$\{x \in X \mid x \in I_{\mathcal{S}}(\{y, z\}) \iff x \in \{y, z\}\}.$$

Let $(X, \mathcal{S})$, $(X', \mathcal{S}')$ be compact Hausdorff spaces with fixed subbases. A map $f\colon X \to X'$ is called *convex* if for every $S' \in \mathcal{S}'$ the set $f^{-1}(S')$ can be represented as the intersection of some subfamily of $\mathcal{S}$.

Lemma 3.6.11. *For every convex map $f\colon (X, \mathcal{S}) \to (X', \mathcal{S}')$ and $A \subset X$ the following holds: $f(I_{\mathcal{S}}(A)) \subset I_{\mathcal{S}'}(f(A))$.* □

Denote by $\mathcal{P}$ the category of couples $(X, \mathcal{S})$, where $\mathcal{S}$ is a binary almost normal T_2-subbase of a compact Hausdorff space$X$, and their convex maps.

Lemma 3.6.12. *Let $(X, \mathcal{S})$ be an object of the category $\mathcal{P}$ and $\mathcal{M} \in \lambda X$. Then $|\bigcap \{I_{\mathcal{S}}(M) \mid M \in \mathcal{M}\}| = 1$.*

Proof. By binarity of $\mathcal{S}$, we have

$$\bigcap \{I_{\mathcal{S}}(M) \mid M \in \mathcal{M}\} \neq \varnothing.$$

Suppose that $x_1, x_2 \in \bigcap \{I_{\mathcal{S}}(M) \mid M \in \mathcal{M}\}$ and $x_1 \neq x_2$. Since $\mathcal{S}$ is a T_2-subbase, there exist sets $S_1, S_2 \in \mathcal{S}$ with $S_1 \cup S_2 = X$, $x_1 \notin S_1$,

$x_2 \notin S_2$. Using the maximality of $\mathcal{M}$, we can suppose that $S_1 \in \mathcal{M}$. Then

$$x_1 \in S_1 = I_S(S_1) \supset \bigcap\{I_S(M) \mid M \in \mathcal{M}\}.$$

Contradiction. □

Applying this lemma, define the map $k_S \colon \lambda X \to X$ putting $k_S(\mathcal{M}) \in \bigcap\{I_S(M) \mid M \in \mathcal{M}\}$, $\mathcal{M} \in \lambda X$.

Lemma 3.6.13. *The map k_S is continuous.*

Proof. Let $x = k_S(\mathcal{M})$ and U be a neighborhood of x (in X). Using almost normality of $\mathcal{S}$, for every $y \in X \setminus U$ choose sets $S_y, S'_y \in \mathcal{S}$ with $S_y \in \mathcal{M}$, $S_y \operatorname{Int} S'_y$, and $y \notin S'_y$. Then $\bigcap\{S'_y \mid y \in X \setminus U\} = \varnothing$ and there exists a finite subset $\{y_1, \dots, y_l\} \subset X \setminus U$ such that

$$\bigcap\{S'_{y_i} \mid 1 \leq i \leq l\} \subset X \setminus U.$$

Thus, $\mathcal{M} \in \bigcap\{(\operatorname{Int} S'_{y_i})^+ \mid 1 \leq i \leq l\}$ and

$$k_S(\bigcap\{(\operatorname{Int} S'_{y_i})^+ \mid 1 \leq i \leq l\}) \subset \bigcap\{S'_{y_i} \mid 1 \leq i \leq l\} \subset U.$$

Finally, note that $\bigcap\{(\operatorname{Int} S'_{y_i})^+ \mid 1 \leq i \leq l\}$ is a neighborhood of $\mathcal{M} \in \lambda X$. □

Lemma 3.6.14. *If (X, ξ) is an $\mathbb{L}$-algebra then $\xi(\xi(A^+)^+) = \xi(A^+)$ for every $A \in \exp(X)$.*

Proof. Consider the set $A^{++} \subset \lambda^2(X)$. We have

$$\xi \circ \lambda\xi(A^{++}) = \xi(\xi(A^+)^+).$$

In the other hand, $\xi \circ \lambda\xi(A^{++}) = \xi \circ \mu X(A^{++})$. Therefore, it is sufficient to show that $\mu X(^{++}) = A^+$.

If $\mathfrak{M} \in A^{++}$ then $A^+ \in \mathfrak{M}$. Thus $A \in \mu X(\mathfrak{M})$, i.e., $\mu X(\mathfrak{M}) \in A^+$. Converserly, if $\mathcal{M} \in A^+$ then $\eta\lambda X(\mathcal{M}) \in A^{++}$ and $\mathcal{M} = \mu X \circ \eta\lambda X(\mathcal{M}) \in \mu X(A^{++})$. □

Theorem 3.6.15. *A couple (X, ξ) is an $\mathbb{L}$-algebra if and only if there exists a binary almost T_2-subbase $\mathcal{S}$ of X such that $\xi = k_S$.*

Proof. Necessity. Let (X,ξ) be an $\mathbb{L}$-algebra. Put $\mathcal{S} = \{\xi(A^+) \mid A \in \exp X\}$. Clearly, $\mathcal{S}$ is a subbase of X. Show the almost normality of $\mathcal{S}$. Let $S = \xi(A^+) \in \mathcal{S}$ and OS be a neighborhood of S. By Lemma 3.6.14 we have $S = \xi(S^+)$. Since the map $(-)^+\colon \exp X \to \exp \lambda X$ is continuous and S belongs to the closure of $\{S' \in \exp X \mid S \subset \operatorname{Int} S'\}$ in $\exp X$, there exists $S' \in \exp X$ such that $\xi\big((S')^+\big) \subset OS$ and $S \subset \operatorname{Int} S'$. Then

$$S \subset \operatorname{Int}\big(\xi\big((S')^+\big)\big) \subset \xi\big((S')^+\big) \subset OS.$$

Now, prove that $\mathcal{S}$ is a T_2-subbase. For this, show that for every $\mathcal{M} \in \lambda X$ the set $K = \bigcap\{\xi(M^{++}) \mid M \in \mathcal{M}\}$ is single. Indeed, it is obvious that $\xi(\mathcal{M}) \in K$. Suppose that there exists $x \in K$ with $x \neq \xi(\mathcal{M})$. For a given $M \in \mathcal{M}$ choose a point $p(M) \in \xi^{-1}(x) \cap M^+$. Let $B = \overline{\{p(M) \mid M \in \mathcal{M}\}}$ and $\mathfrak{M} \in \lambda^2 X$ be a unique maximal chained system containning the chained system $\{B\} \cup \{\{\mathcal{M}, p(M)\} \mid M \in \mathcal{M}\}$. Then

$$\xi \circ \lambda\xi(\mathfrak{M}) = \xi \circ \eta X\left(\xi(B)\right) = \xi \circ \eta X\left(x\right) = x.$$

But

$$\begin{aligned}
&\xi \circ \lambda\xi(\mathfrak{M}) = \\
&\quad = \xi \circ \mu X(\mathfrak{M}) \in \xi\big(\bigcap\{I_{\mathcal{L}(X)}(\{\mathcal{M}, p(M)\}) \mid M \in \mathcal{M}\} \cap I_{\mathcal{L}(X)}(B)\big) \subset \\
&\quad \subset \xi\big(\bigcap\{M^+ \mid M \in \mathcal{M}\big) = \{\xi(\mathcal{M})\}.
\end{aligned}$$

A contradiction.

Suppose that there exist points $x_1 \neq x_2$ of X, separating no closures of elements of $\mathcal{S}$. Consider the chained system

$$\mathcal{M} = \big\{\{x_1, x_2\}\big\} \cup \big\{\overline{X \setminus S} \mid |\overline{X \setminus S} \cap \{x_1, x_2\}| = 1,\ S \in \mathcal{S}\big\}$$

and let $\mathcal{M} \subset \mathcal{M}' \in \lambda X$. Show that $I_{\mathcal{S}}(M) \supset \{x_1, x_2\}$ for every $M \in \mathcal{M}$. Indeed, we can suppose that $x_1 \in M$. If $x_2 \notin I_{\mathcal{S}}(M)$ then there exists $S \in \mathcal{S}$ such that $M \subset S$ and $x_2 \notin S$. Choose $S' \in \mathcal{S}$ with $S \subset \operatorname{Int}(S')$ and $x_2 \notin S$. Thus, $X \setminus \operatorname{Int}(S') \in \mathcal{M}$ and $M \cap (X \setminus \operatorname{Int}(S')) = \varnothing$ and we obtain a contradiction with chainity of $\mathcal{M}$. Hence, we have $I_{\mathcal{S}}(M) \supset \{x_1, x_2\}$ for every $M \in \mathcal{M}$. Note that $I_{\mathcal{S}}(M) \subset \xi(M^+)$. Therefore,

$$\{x_1, x_2\} \subset \bigcap\{I_{\mathcal{S}}(M) \mid M \in \mathcal{M}\} \subset \bigcap\{\xi(M^+) \mid M \in \mathcal{M}\}.$$

A contradiction.

Suficity. Let S be a binary almost normal T_2-subbase of X and $\xi = k_S\colon \lambda X \to X$. Clearly, $\xi \circ \eta X = 1_X$. Now it is suficient to show that $\xi \circ \lambda\xi = \xi \circ \mu X$.

Let $\mathfrak{M} \in \lambda^2 X$. Then

$$\begin{aligned}\xi \circ \mu X(\mathfrak{M}) &= \xi(\{M \mid M \in \bigcap \mathcal{A}, \mathcal{A} \in \mathfrak{M}\}) = \\ &= \bigcap\{I_S(M) \mid M \in \bigcap \mathcal{A}, \mathcal{A} \in \mathfrak{M}\}.\end{aligned}$$

In the other hand,

$$\xi \circ \lambda\xi(\mathfrak{M}) = \xi(\{\xi(\mathcal{A}) \mid \mathcal{A} \in \mathfrak{M}\}) \in \bigcap\{I_S(\xi(\mathcal{A})) \mid \mathcal{A} \in \mathfrak{M}\}.$$

Since S is binary, for $\xi \circ \mu X(\mathfrak{M}) = \xi \circ \lambda\xi(\mathfrak{M})$ it is sufficient to prove that

$$I_S(M) \cap I_S(\xi(\mathcal{A}')) \neq \varnothing$$

for all $M, \mathcal{A}, \mathcal{A}'$ with $\mathcal{A}, \mathcal{A}' \in \mathfrak{M}$, $M \in \bigcap \mathcal{A}$. For this we need only $I_S(M) \cap \xi(\mathcal{A}') \neq \varnothing$. Let $\mathcal{M} \in \mathcal{A} \cap \mathcal{A}'$. Then $\xi(\mathcal{M}) \in \xi(\mathcal{A}')$ and

$$\xi(\mathcal{M}) \in \bigcap\{I_S(N) \mid N \in \mathcal{M}\} \subset I_S(M),$$

because $M \in \mathcal{M}$. □

Below we will also use the notation (X, k_S), where S is the subbase from the previous theorem.

Theorem 3.6.16. *A map $f\colon X \to X'$ is a morphism of $\mathbb{L}$ algebras (X, k_S) and $(X', k_{S'})$ if and only if $f\colon (X, k_S) \to (X', k_{S'})$ is a convex map.*

Proof. Sufficiency. Let $\mathcal{M} \in \lambda X$. Then

$$f \circ k_S(\mathcal{M}) \in f(\bigcap\{I_S(M) \mid M \in \mathcal{M}\})$$

and

$$\begin{aligned}k_{S'}\lambda f(\mathcal{M}) &\in \bigcap\{I_{S'}(N) \mid N \in \lambda f(\mathcal{M})\} = \bigcap\{I_{S'}(f(M)) \mid M \in \mathcal{M}\} \supset \\ &\supset \bigcap\{f(I_S(M)) \mid M \in \mathcal{M}\} \supset f(\bigcap\{I_S(M) \mid M \in \mathcal{M}\}) = \\ &= \{f(k_S(\mathcal{M}))\}.\end{aligned}$$

Hence, $k_{S'} \circ \lambda f = f \circ k_S$, i.e., f is a monad morphism.

Necessity. Suppose that f is not convex. Thus, there exists $M \in \mathcal{S}'$ such that the set $f^{-1}(M)$ cannot be represented as the intersection of some subfamily of $\mathcal{S}$, i.e., $f^{-1}(M) \neq I_{\mathcal{S}}(f^{-1}(M))$. Let $x \in I_{\mathcal{S}}(f^{-1}(M)) \setminus f^{-1}(M)$. There exists $\mathcal{M} \in \lambda X$ containning the following chained system

$$\{f^{-1}(M)\} \cup \{\{x, y\} \mid y \in f^{-1}(M)\}.$$

Then $k_{\mathcal{S}}(\mathcal{M}) \in \bigcap I_{\mathcal{S}}(A) \mid A \in \mathcal{M}\} \supset \{x\}$. Hence, $k_{\mathcal{S}}(\mathcal{M}) = \{x\}$. In the other hand,

$$k_{\mathcal{S}'} \circ \lambda f(\mathcal{M}) \in I_{\mathcal{S}}(f(f^{-1}(M))) \subset I_{\mathcal{S}'}(M) = M.$$

But $f \circ k_{\mathcal{S}}(\mathcal{M}) = f(x) \notin \mathcal{M}$. A contradiction. □

Corollary 3.6.17. *The categories* $\mathbf{Comp}^{\mathbb{L}}$ *and* $\mathcal{P}$ *are isomorphic.* □

Below we characterize the free $\mathbb{L}$-algebras. Any $\mathbb{L}$-algebra of the form $(\lambda X, \mu X)$ is called a *superextension.*

Lemma 3.6.18. *For every* X *we have*

$$q_{\mathcal{L}(X)}(\lambda X) = \eta X.$$

Proof. Let $x \in X$ and $\eta X(x) \in I_{\mathcal{L}(X)}(\{\mathcal{M}, \mathcal{N}\})$ for some $\mathcal{M}, \mathcal{N} \in \lambda X$. Then $\{x\} \in \mathcal{M} \cap \mathcal{N}$ and hence, $\mathcal{M} = \mathcal{N} = \eta X(x)$ and $\eta X(X) \subset q_{\mathcal{L}(X)}(\lambda X)$. On the other hand, if $\mathcal{M} \in \lambda X \setminus \eta X(X)$ then there exists a minimal by the inclusion relation element $M \in \mathcal{M}$ with $|M| \geq 2$. Leet $N_1, N_2 \in \exp X$ be such that $N_1 \cup N_2 = M$ and $N_1 \neq M \neq N_2$.

It is not hard to show that there exists a unique maximal linked system $\mathcal{N}_i$, containing the linked system

$$\{N_i\} \cup \{M \in \mathcal{M} \mid M \cap N_i \neq \varnothing\}, \quad i = 1, 2.$$

(see M. Van de Vel [1979]). Thus evidently, $|\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}\}| = 3$. Moreover,

$$I_{\mathcal{L}(X)}(\{\mathcal{N}_1, \mathcal{N}_2\}) = \bigcap\{A^+ \mid A \in \exp X,\ A \in \mathcal{N}_1 \cap \mathcal{N}_2\}.$$

This fact and closedness of $\eta X(X)$ in λX imply the statement of lemma. □

Theorem 3.6.19. *Let (X, ξ) (or (X, S), where $\xi = k_S$) be an $\mathbb{L}$-algebra. Then the following conditions are equivalent:*
1) *(X, ξ) is a superextension;*
2) *(X, ξ) is a free $\mathbb{L}$-algebra;*
3) *(X, ξ) is freely generated by the set $q_S(X)$;*
4) *(X, ξ) is isomorphic to $(\lambda(q_S(X)), \mu q_S(X))$.*

Proof. We have the implication 4)$\Rightarrow$1) by the definition. 1)$\Rightarrow$2) is a well known general fact (MacLane [1971]).

2)$\Rightarrow$3). Let a couple (Z, i) freely generate the $\mathbb{L}$-algebra (X, ξ), where $i\colon Z \to X$ is a map. Then for every $\mathbb{L}$-algebra (X', ξ') and map $f\colon Z \to X'$ there exists a unique morphism $g\colon (X, \xi) \to (X', \xi')$ of $\mathbb{L}$-algebras such that $f = g \circ i$. Since the couple $(Z, \eta Z)$ freely generates the $\mathbb{L}$-algebra $(\lambda Z, \mu Z)$, there exists a unique morphism $h\colon (\lambda Z, \mu Z) \to (X, \xi)$ of $\mathbb{L}$-algebras such that $i = h \circ \eta Z$. On the other hand, there exists a unique $k\colon (X, \xi) \to (\lambda Z, \mu Z)$ of $\mathbb{L}$-algebras with $\eta Z = k \circ i$. Consequently, k and h are homeomorphisms. We obtain that i is an embedding. By Theorem 3.6.16, the maps k and h are convex. Show that $i(Z) = q_S(X)$. Let $z \in Z$ and $i(z) \notin q_S(X)$. Then there exist points $x, y \in X$ with $i(z) \notin \{x, y\}$ and $i(z) \in I_S(\{x, y\})$. Therefore, $\eta Z \notin \{k(x), k(y)\}$. But by Lemma 3.6.11,

$$k(I_S(\{x, y\})) \subset I_{\mathcal{L}(Z)}(\{k(x), k(y)\}),$$

and thus,

$$\eta Z(z) = k \circ i(z) \in I_{\mathcal{L}(Z)}(\{k(x), k(y)\}).$$

Since this contradicts with Lemma 3.6.18. Hence, $i(Z) \subset q_S(X)$. One can similarly prove the inverse inclusion. Therefore, the $\mathbb{L}$-algebra (X, ξ) is freely generated by $q_S(X)$.

The proving of 3)$\Rightarrow$4) is similar. □

3.6.4. Algebras of the inclusion hyperspace monad

Definition 3.6.20. A binary closed subbase $\mathcal{S}$ of a compact Hausdorff space X is called *bisupercompact* if it has a representation $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$ for which

a) the subsystems $\mathcal{S}^+$ and $\mathcal{S}^-$ are linked;

b) for every distinct $x_1, x_2 \in X$ there exist elements $S_1 \in \mathcal{S}^+$ and $\S_2 \in \mathcal{S}^-$ such that $S_1 \cup S_2 = X$ ans the sets $X \setminus S_1, X \setminus S_2$ separates the points x_1 and x_2;

c) for every element $S \in \mathcal{S}^+$ ($S \in C\mathcal{S}^-$) and neighborhood $OS \supset S$ there exists an element $S' \in \mathcal{S}^+$ (respectively, $S' \in \mathcal{S}^-$) with $S \subset \operatorname{Int} S' \subset S' \subset OS$.

For every $A \in \exp X$ put

$$\begin{aligned} A^+ &= \{\mathcal{A} \in GX \mid A \in \mathcal{A}\}, \\ A^- &= \{\mathcal{A} \in GX \mid B \cap A \neq \varnothing \text{ for all } B \in \mathcal{A}\}. \end{aligned}$$

One can show that the system

$$\mathcal{L}(X) = \{A^+ \mid A \in \exp X\} \cup \{A^- \mid A \in \exp X\}$$

is a bisupercompact subbase.

Recalling the previous subsection, we can easily prove the following results.

Lemma 3.6.21. *Let $(X, \mathcal{S})$ be a space with a supercompact subbase. Then for every $\mathcal{A} \in GX$ the set*

$$\begin{aligned} K_{\mathcal{S}}(\mathcal{A}) = \bigcap\{S \in \mathcal{S}^+ \mid S \supset A \text{ for some } A \in \mathcal{A}\} \cap \\ \cap \bigcap\{S \in \mathcal{S}^- \mid S \cap B \neq \varnothing \text{ for all } B \in \mathcal{A}\} \end{aligned}$$

is single.

By this lemma, we define the map $k_{\mathcal{S}} \colon GX \to X$ by $k_{\mathcal{S}}(\mathcal{A}) \in K_{\mathcal{S}}(\mathcal{A})$.

Lemma 3.6.22. *The map $k_{\mathcal{S}}$ is continuous.*

Lemma 3.6.23. *If (X, ξ) is a G-algebra, then for every $A \in \exp X$ we have $\xi(\xi(A^+)^+) = \xi(A^+)$ and $\xi(\xi(A^-)^-) = \xi(A^-)$.*

Lemma 3.6.24. *If (X, ξ) is a G-algebra and $\mathcal{A}, \mathcal{B} \in GX$, then the following equalities*

$$\begin{aligned} \xi(\mathcal{A} \cap \mathcal{B}) &= \xi(\eta X \circ \xi(\mathcal{A}) \cap \eta X \circ \xi(\mathcal{B})), \\ \xi(\mathcal{A} \cup \mathcal{B}) &= \xi(\eta X \circ \xi(\mathcal{A}) \cup \eta X \circ \xi(\mathcal{B})) \end{aligned}$$

hold.

Proof. Show only the first equality. Put $\mathfrak{A} = \{\mathcal{M} \in \exp GX \mid \{\mathcal{A}, \mathcal{B}\} \subset \mathcal{M}\}$. Obviously, $\mu X(\mathfrak{A}) = \mathcal{A} \cap \mathcal{B}$ and $G\xi(\mathfrak{A}) = \eta X \circ \xi(\mathcal{A}) \cap \eta X \circ \xi(\mathcal{B})$. We have only to note that $\xi \circ \mu X = \xi \circ G\xi$. $\square$

A compact Hausdorff lattice is called a *Lawson lattice* if it admits a separating family of continuous lattice homomorphisms into the lattice $[0,1]$ with the operations max and min.

Theorem 3.6.25. *The following conditions are equivalent:*
1) A couple (X,ξ) is a G-algebra;
2) X is a Lawson lattice with $\xi(\mathcal{A}) = \sup\{\inf A \mid A \in \mathcal{A}\}$, $\mathcal{A} \in GX$.
3) There exists a bisupercompact subbase $\mathcal{S}$ of X such that $\xi = k_{\mathcal{S}}$.

Proof. 1)$\Rightarrow$2). Define binary operations $\wedge$ and $\vee$ on X, setting

$$x \wedge y = \xi\big(\eta X(x) \cap \eta X(y)\big),$$
$$x \vee y = \xi\big(\eta X(x) \cup \eta X(y)\big).$$

It is easy to see that these operations form a distributive compact lattice structure on X. Moreover, for every $A \in \exp X$ we have

$$\inf(A) = \xi(\{B \in \exp X \mid B \supset A\}),$$
$$\sup(A) = \xi(\{B \in \exp X \mid B \cap A \neq \varnothing\}).$$

Thus, the maps $\sup, \inf\colon \exp X \to X$ are continuous, i.e., X is a Lawson lattice. One can directly verify the equality $\xi(\mathcal{A}) = \sup\{\inf A \mid A \in \mathcal{A}\}$.

2)$\Rightarrow$1). Let X be a Lawson lattice, and $\xi(\mathcal{A}) = \sup\{\inf A \mid A \in \mathcal{A}\}$ for all $\mathcal{A} \in GX$. Clearly, $\xi \circ \eta X = 1_X$. Show that $\xi \circ \mu X = \xi \circ G\xi$. For every $\mathfrak{A} \in G^2X$ we have

$$\begin{aligned}
\xi \circ G\xi(\mathfrak{A}) &= \xi \circ rX \circ \exp^2 \xi(\mathfrak{A})\\
&= \xi \circ rX\big(\{\{\xi(\mathcal{A}) \mid \mathcal{A} \in \mathcal{M}\} \mid \mathcal{M} \in \mathfrak{A}\}\big)\\
&= \xi\big(\{B \in \exp X \mid B \supset \{\sup\{\inf A \mid A \in \mathcal{A}\} \mid \mathcal{A} \in \mathcal{M}\},\ \mathcal{M} \in \mathfrak{A}\}\big)\\
&= \sup\big\{\inf\big\{\sup\{\inf A \mid A \in \mathcal{A}\} \mid \mathcal{A} \in \mathcal{M}\big\},\ \mathcal{M} \in \mathfrak{A}\big\}
\end{aligned}$$

(here $rX\colon \exp^2 X \to GX$ is the natural retraction $rX(\mathcal{A}) = \{B \in \exp X \mid B \supset A$ for some $A \in \mathcal{A}\}$). For each $\mathcal{M} \in \mathfrak{A}$ denote by $F_{\mathcal{M}}$ the set of all (not necessarily continuous) maps $\varphi\colon \mathcal{M} \to \bigcup \mathcal{M}$ such that

$\varphi(\mathcal{A}) \in C\mathcal{A}$. Then

$$\begin{aligned}\xi \circ \mu X(\mathfrak{A}) &= \xi(\{A \mid A \in \bigcap\mathcal{M},\ \mathcal{M} \in \mathfrak{A}\}) \\ &= \sup\{\inf A \mid A \in \bigcap\mathcal{M},\ \mathcal{M} \in \mathfrak{A}\} \\ &= \sup\{\sup\{\inf A \mid A \in \bigcap\mathcal{M}\} \mid \mathcal{M} \in \mathfrak{A}\} \\ &= \sup\{\sup\{\inf(\bigcup\{\varphi(\mathcal{A}) \mid \mathcal{A} \in \mathcal{M}\}) \mid \varphi \in F_{\mathcal{M}}\} \mid \mathcal{M} \in \mathfrak{A}\} \\ &= \{\sup\{\inf\{\sup\{\inf A \mid A \in \mathcal{A}\} \mid \mathcal{A} \in \mathcal{M}\} \mid \mathcal{M} \in \mathfrak{A}\}\} \\ &= \xi \circ G\xi(\mathfrak{A})\end{aligned}$$

(here the fifth equality follows from complete distributivity of the lattice X).

3)$\Rightarrow$1). Let (X, ξ) be a G-algebra. Put

$$\begin{aligned}\mathcal{S}^- &= \{\xi(A^-) \mid A \in \exp X\}, \\ \mathcal{S}^+ &= \{\xi(A^+) \mid A \in \exp X\}.\end{aligned}$$

Clearly, $\mathcal{S} = \mathcal{S}^- \cup \mathcal{S}^+$ is a subbase of X. By Lemma 3.6.23 and bisupercompactness of $\mathcal{L}(X)$ we have that the subbase $\mathcal{S}$ is binary and the systems $\mathcal{S}^-$ and $\mathcal{S}^+$ are linked. Using Lemma 3.6.23, one can prove property c) of bisupercompact subbase for $\mathcal{S}$ similary to proving the almost normality property of the respective subbase in Theorem 3.6.15.

Show property b) of bisupercompact subbase for $\mathcal{S}$. For this prove that the set

$$K = \bigcap(\{\xi(M^+) \mid M \in \mathcal{A}\} \cup \{\xi(M^-) \mid M \cap A \neq \varnothing \text{ for all } A \in \mathcal{A}\})$$

is single for every $\mathcal{A} \in GX$. Note that $\xi(\mathcal{A}) \in K$ and suppose that there exists a point $x \in K$ with $x \neq \xi(\mathcal{A})$. For every $M \in \mathcal{A}$ choose a point $p(M) \in \xi^{-1}(x) \cap M^+$. Let a set $N \in \exp X$ intersect all elements of the system $\mathcal{A}$. Consider a point $p'(N) \in \xi^{-1}(x) \cap N^-$. Let β be a closure of

$$\{p(M) \mid M \in \mathcal{A}\} \cup \{p'(N) \mid N \cap A \neq \varnothing \text{ for all } A \in \mathcal{A}\}.$$

Define an element $\mathfrak{A} \in G^2X$ by the formula

$$\begin{aligned}\mathfrak{A} = rGX\big(\{\beta\} \cup \{\{\mathcal{A}, p(M)\} \mid M \in \mathcal{A}\} \cup \\ \cup \{\{\mathcal{A}, p'(N)\} \mid N \cap A \neq \varnothing \text{ for every } A \in \mathcal{A}\}\big).\end{aligned}$$

Then

$$\xi \circ G\xi(\mathfrak{A}) = \xi \circ \eta X(\xi(\beta)) = \xi \circ \eta X(x) = x.$$

It is easy to verify that

$$\mu X(\mathfrak{A}) \in \bigcap\{A^+ \mid A^+ \in \mathfrak{A}\} \cap \\ \cap \bigcap\{A^- \mid A \in \exp X,\ A^- \cap \alpha \neq \varnothing \text{ for all } \alpha \in \mathfrak{A}\}$$

and

$$\begin{aligned}&\xi \circ \mu X(\mathfrak{A}) \in \xi\Big(\bigcap\{A^+ \mid A \in \exp X,\ A^+ \supset \beta\}\\ &\cap \bigcap\{A^+ \mid A \in \exp X,\ A^+ \supset \{p(M), \mathcal{A}\} \text{ for some } M \in \mathcal{A}\}\\ &\cap \bigcap\{A^+ \mid A \in \exp X,\ A^+ \supset \{p'(N), \mathcal{A}\} \text{ for some } N \in \exp X,\\ &\text{intersecting all elements of } \mathcal{A}\}\\ &\cap \bigcap\{A^- \mid A \in \exp X,\ p(M) \in A^- \text{ for every } M \in \mathcal{A} \text{ and}\\ &p'(N) \in A^- \text{ for every } N \in \exp X, \text{ intersecting all elements of } \mathcal{A}\}\\ &\cap \bigcap\{A^- \mid A \in \exp X,\ \mathcal{A} \in A^-\}\Big) \in\\ &\in \xi\Big(\bigcap\{A^= \mid \mathcal{A} \in A^+\} \cap \bigcap\{A^- \mid \mathcal{A} \in A^-\}\Big) =\\ &= \xi(\mathcal{A}),\end{aligned}$$

i.e., $\xi(\mathcal{A}) = x$.

Now let there exist points $x_1, x_2 \in X$, which are not separated by complements of elements of $\mathcal{S}$. Put

$$\mathcal{A} = rX\left(\{\{x_1, x_2\}\} \cup \{\overline{X \setminus S} \mid |\overline{X \setminus S} \cap \{x_1, x_2\}| = 1,\ S \in \mathcal{S}\}\right).$$

Using the fact that $\mathcal{A}$ is linked and property c) of $\mathcal{S}$, one can show that the points x_1 and x_2 belong to $K(\mathcal{A})$. Thus

$$\begin{aligned}\{x_1, x_2\} &\subset \bigcap\{S \in \mathcal{S}^+ \mid S \in \mathcal{A}\} \cap \bigcap\{S \in \mathcal{S}^- \mid S \cap A \neq \varnothing \text{ for all } A \in \mathcal{A}\}\\ &\subset \bigcap\{\xi(A^+) \mid A \in \mathcal{A}\}\\ &\quad \cap \bigcap\{\xi(B^-) \mid B \in \exp X \text{ and } A \cap B \neq \varnothing \text{ for every } A \in \mathcal{A}\}\\ &= K(\mathcal{A}).\end{aligned}$$

This contradicts with the proved above facts.

It is easy to check that the points can be only separated by complements of elements of distinct subfamilies of $\mathcal{S}^+$ and $\mathcal{S}^-$. Therefore

$$\xi(\mathcal{A}) = \bigcap\{A \in \mathcal{S}^+ \mid A \in \mathcal{A}\} \cap \bigcap\{B \in \mathcal{S}^- \mid B \cap A \neq \varnothing \text{ for all } A \in \mathcal{A}\}.$$

□

Theorem 3.6.26. *Let (X, ξ) and (X', ξ') be G-algebras. Then the following conditions are equivalent:*
1) $f\colon (X, \xi) \to (X', \xi')$ is a morphism of G-algebras;
2) f is a complete homomorphism of the lattices X and X';
3) $f\colon (X, \mathcal{S}) \to (X', \mathcal{S}')$ is a biconvex map.

Proof. 1)→2). Let $f\colon (X, \xi) \to (X', \xi')$ be a morphism of G-algebras and $C \in \exp X$. Then

$$\begin{aligned} f(\inf C) &= f \circ \xi(\{B \in \exp X \mid B \supset\}) = \\ &= \xi'(\{B \in \exp X' \mid B \supset f(C)\}) = \inf(f(C)). \end{aligned}$$

One can argue similary for the operation sup.

2)⇒1). If $f\colon X \to X'$ is a complete lattice homomorphism, then we have

$$f \circ \xi(\mathcal{A}) = f(\sup\{\inf A \mid A \in \mathcal{A}\}) = \sup\{\inf f(A) \mid A \in \mathcal{A}\} = \xi' \circ Gf(\mathcal{A})$$

for every $\mathcal{A} \in GX$, i.e., f is a morphism of G-algebras.

1)⇒3). Let $f\colon (X, \xi) \to (X', \xi')$ be a morphism of G-algebras. For every $A \subset X$ set

$$I_{\mathcal{S}}^{\pm}(A) = \bigcap\{S \in \mathcal{S}^{\pm} \mid S \supset A\}.$$

Suppose that the map f is not biconvex and $f^{-1}(T) \neq I_{\mathcal{S}}^{+}(f^{-1}(T))$, where $T \in \mathcal{S}'^{+}$. Choose $a \in I_{\mathcal{S}}^{+}(f^{-1}(T)) \setminus f^{-1}(T)$ and put

$$\mathcal{A} = \{A \in \exp X \mid a \in A \text{ or } f^{-1}(T) \subset A\}.$$

Then

$$\begin{aligned} \xi(\mathcal{A}) = &\bigcap\{A \in \mathcal{S}^{+} \mid a \in A\} \cap \bigcap\{A \in \mathcal{S}^{+} \mid A \supset f^{-1}(T)\} \cap \\ &\cap \bigcap\{A \in \mathcal{S}^{-} \mid a \in A \text{ and } A \cap f^{-1}(T) \neq \varnothing\}. \end{aligned}$$

Since $a \in I_{\mathcal{S}}^{+}(f^{-1}(T))$, we have $\xi(\mathcal{A}) = a$. But on the other hand,

$$\xi' \circ Gf(\mathcal{A}) \in I_{\mathcal{S}'}^{+}(f \circ f^{-1}(T)) = I_{\mathcal{S}'}^{+}(T) = T,$$

i.e., $a \in f^{-1}(T)$. A contradiction. The case of $T \in \mathcal{S}'^{-}$ is similar.

3)⇒1). Let $f\colon (X,\mathcal{S}) \to (X',\mathcal{S}')$ be a biconvex map and $\mathcal{A} \in GX$. Then

$$f \circ \xi(\mathcal{A}) = f(\bigcap\{S \in \mathcal{S}^+ \mid S \supset A \text{ for some } A \in \mathcal{A}\} \cap \\ \cap \bigcap\{S \in \mathcal{S}^- \mid S \cap A \neq \varnothing \text{ for all } A \in \mathcal{A}\}).$$

On the other hand,

$$\xi' \circ Gf(\mathcal{A}) = \bigcap\{S \in \mathcal{S}'^+ \mid S \supset f(A) \text{ for some } A \in \mathcal{A}\} \cap \\ \cap \bigcap\{S \in \mathcal{S}'^- \mid S \cap f(A) \neq \varnothing \text{ for every } A \in \mathcal{A}\}.$$

Since f is a biconwex map, for every $S \in \mathcal{S}^+$, containing $f(A)$ for some $A \in \mathcal{A}$, we have $f^{-1}(S) = \bigcap \mathcal{L}$, where $\mathcal{L} \subset \mathcal{S}^+$ and every element of $\mathcal{L}$ contains the set A. Hence, $S = \bigcap\{f(L) \mid L \in \mathcal{L},\ L \supset A\}$. Similary, for every $S \in \mathcal{S}'^-$ we have

$$S = \bigcap\{f(L) \mid L \in \mathcal{L}' \text{ and } L \cap A \neq \varnothing \text{ for all } A \in \mathcal{A}\},$$

where $\mathcal{L}' \subset \mathcal{S}^-$. Obtain that

$$\xi' \circ Gf(\mathcal{A}) = \bigcap\{f(S) \mid S \in \mathcal{S}^+,\ S \supset f(A) \text{ for some } A \in \mathcal{A}\} \cap \\ \cap \bigcap\{f(S) \mid S \in \mathcal{S}^-,\ S \cap A \neq \varnothing \text{ for every } A \in \mathcal{A}\} = \\ = f \circ (\mathcal{A}).$$

□

Free G-algebras. For every lattices X we put

$$qX = \{x \in X \mid \text{if } x = \sup\{y,z\} \text{ then } x \in \{y,z\} \text{ and} \\ \text{if } x = \inf\{y_1,z_1\} \text{ then } x \in \{y_1,z_1\}\}$$

Lemma 3.6.27. *For every X we have $qGX = \eta X(X)$.*

Proof. Let $a \in X$ and $\eta X(a) = \mathcal{B}_1 \cup \mathcal{B}_2$. If $\{a\} \in \mathcal{B}_i$ then $\eta X(a) = \mathcal{B}_i$, $i = 1,2$. Next let $\eta X(a) = \mathcal{B}_1 \cap \mathcal{B}_2$. Then $\{a\} \in \mathcal{B}_1 \cap \mathcal{B}_2$. Supposing that the family $\mathcal{B}_i$ contains an element R with $a \notin R$, we obtain that every element of $\mathcal{B}_j$ contains $\{a\}$ (here, $\{i,j\} = \{1,2\}$). Hence, $\mathcal{B}_j = \eta X(a)$ and $\eta X(X) \subset qGX$.

Show the inverse inclusion. Let $\mathcal{A} \in qGX \setminus \eta X(X)$. Suppose firstly that $\mathcal{A}$ contains a non-single minimal by inclusion set A. Put $A = A_1 \cup A_2$, where A_1, A_2 are closed sets with $A_1 \neq A \neq A_2$. Set

$$\mathcal{A}_i = \mathcal{A} \cup rX(\{A_i\}).$$

Then $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A} \notin \{\mathcal{A}_1, \mathcal{A}_2\}$. Now let every minimal by inclusion element of $\mathcal{A}$ be single. We denote by B the union of such elements. Present B as $B = B_1 \cup B_2$, where B_1, B_2 are closed sets with $B_1 \neq B \neq B_2$. Put

$$\mathcal{B}_i = \{C \in \exp X \mid C \cap B_i \neq \varnothing\}.$$

Then $\mathcal{A} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{A} \notin \{\mathcal{B}_1, \mathcal{B}_2\}$. □

Theorem 3.6.28. *Let (X,ξ) be a G-algebra. The following conditions are equivalent:*
1) (X,ξ) is a free G-algebra;
2) (X,ξ) is freely generated by subspace qX;
3) (X,ξ) is isomorphic to $(GqX, \mu qX)$.

The reader can prove this fact similarly to Theorem 3.6.19. He have only to use Lemma 3.6.27 instead of Lemma 3.6.18.

Here we give a criterion of openness of the structure maps of G-algebras. Let X be a Lawson semilattice. For $k \in \mathbb{N}$, define the maps $i_k, s_k : X \to \exp_k(\exp X)$ by the formulae:

$$i_k(x) = \{A \in \exp_k X \mid \inf A = x\}, \quad s_k(x) = \{A \in \exp_k X \mid \sup A = x\}.$$

Theorem 3.6.29. *Let (X,ξ) be a $\mathbb{G}$-algebra. The structure map $\xi\colon GX \to X$ is open if and only if the maps i_k, s_k are continuous for every $k \in \mathbb{N}$.*

Proof. Necessity. Suppose ξ is open, while the map i_k is not continuous, for some $k \in \mathbb{N}$. Then there exist $x \in X$, $\{x_1, \dots, x_k\} \in \exp_k X$, a net $(x_\alpha)_{\alpha \in \mathfrak{A}}$, and open in X sets $V_1, \dots, V_k$ satisfying the conditions:

1) $(x_\alpha)_{\alpha \in \mathfrak{A}} \to x$;

2) $x = \inf\{x_1, \dots, x_k\}$;

3) V_i are dijoint neighborhoods of x_i, $1 \leq i \leq k$;

4) if $\alpha \in \mathfrak{A}$, there is no $A \in \exp_k X \cap \langle V_1, \dots, V_k \rangle$ such that $x_\alpha = \inf A$.

To obtain a contradiction, consider

$$\mathcal{A} = \{A \in \exp X \mid A \supset \{x_1, \dots, x_k\}\} \in GX$$

and the neighborhood

$$O = \langle \langle U_1, \dots, U_k \rangle, \langle U_1, \dots, U_k, X \rangle,$$

where $U_i \subset V_i$ is a closed neighborhood of x_i, $1 \le i \le k$. Since X is a Lawson lattice, we may suppose that $U_i = \{c \in X \mid b_i \le c \le a_i$, for some $a_i, b_i \in X$.

Now, let $\mathcal{A}' \in O$ and $a' = \xi(\mathcal{A}') = \sup\{\inf A' \mid A' \in \mathcal{A}'\}$. There exists $A_0 \in \mathcal{A}'$, $A_0 \in \langle U_1, \dots, U_k \rangle$, and, therefore, there exists $d_i \in U_i$ such that $\inf\{d_1, \dots, d_k\} = \inf A_0$. Consequently, $a' \ge \inf\{b_1, \dots, b_k\}$. On the other hand, since every element of $\mathcal{A}'$ meets every U_i, we see that $\inf A' \le \inf\{a_1, \dots, a_k\}$ for every $A' \in \mathcal{A}'$, and hence $a' \le \{a_1, \dots, a_k\}$. Since

$$\inf\{b_1, \dots, b_k\} \le a' \le \{a_1, \dots, a_k\},$$

there exist $c_i \in U_i \subset V_i$ such that $a' = \inf\{c_1, \dots, c_k\}$. This contradiction shows that $\xi^{-1}(a_\alpha) \cap O = \varnothing$, and, therefore, the map ξ is not open.

Analogous arguments show that if the map s_k is not continuous, the map ξ is not open.

Sufficiency. Show that continuity of the map i_k implies openness of the map $\inf\colon \exp X \to X$. Let $A \in \exp X$, $a = \inf A$, and $\langle V_1, \dots, V_n \rangle$ be a neighborhood of A in $\exp X$. Choose a neighborhood $\langle U_1, \dots, U_l \rangle \subset \langle V_1, \dots, V_n \rangle$, where $U_i = \operatorname{int}\{c \in X \mid a_i \le c \le b_i\}$, for $a_i, b_i \in X$. Then for every $i \in \{1, \dots, l\}$ there exists $c_i \in U_i$ such that $a = \inf\{c_1, \dots, c_l\}$. Since $\{c_1, \dots, c_l\} \in \langle V_1, \dots, V_n \rangle$ and the map i_l is continuous, There exists a neighborhood O of a in X such that for every $a' \in O$ there exists $B \in \exp_l X$ such that $B \in \langle V_1, \dots, V_n \rangle$ and $\inf B = a'$. This shows that the map inf is open. Similarly, it can be proved that continuity of maps s_k implies openness of the map sup $\exp X \to X$. Since the functor exp is open, the map $\xi = \sup \circ \exp \inf$ is open.

□

3.6.5. Algebras of the monads $\tilde{\mathbb{G}}$ and $\tilde{\mathbb{N}}$

Theorem 3.6.30. *A couple (X,ξ) is a $\tilde{\mathbb{G}}$-algebra iff there exist two jointly completely distributive Lawson semilattices structures on X such that*
$\xi(\mathcal{A}) = \vee\{\wedge A \mid A \in \mathcal{A}\}$ for every $\mathcal{A} \in \tilde{G}X$.

Proof. Necessity. Let (X,ξ) be a $\tilde{\mathbb{G}}$-algebra and $\mathcal{A},\mathcal{B} \in \tilde{G}X$. Show that

$$\xi(\mathcal{A}\tilde{\cap}\mathcal{B}) = \xi(\{\{\xi(\mathcal{A})\}\}\tilde{\cap}\{\{\xi\mathcal{B})\}\}).$$

Suppose that $\xi(\mathcal{A}) = a$, $\xi(\mathcal{B}) = b$. Consider $\mathfrak{A} = \{\{\mathcal{A},\mathcal{B}\}\} \in \tilde{G}X$. Then $\mu X(\mathfrak{A}) = \mathcal{A}\tilde{\cap}\mathcal{B}$ and

$$\tilde{G}(\xi)(\mathfrak{A}) = \{\{a,b\}\} = \{\{a\}\}\tilde{\cap}\{\{b\}\}.$$

Hence,

$$\xi(\mathcal{A}\tilde{\cap}\mathcal{B}) = \xi \circ \mu X(\mathfrak{A}) = \xi \circ \tilde{G}(\xi)(\mathfrak{A}) = \xi(\{\{a\}\}\tilde{\cap}\{\{b\}\}.$$

Now one can well define a semilattice operation $\wedge$ by the formula:

$$\wedge A = \xi(\tilde{\bigcap}\{\{\{a\}\} \mid a \in A\})$$

for every $A \in \exp X$. It is easy to verify directly the semilattice axioms. Obviously, $\wedge\colon \exp X \to X$ is continuous. Hence, the semilattice $(X,\wedge)$ is the Lawson one.

The second semilattice structure on X (with an operation $\vee$) can be similarly constructed by $\tilde{\cup}$. Since $\tilde{\cap}$ and $\tilde{\cup}$ are completely distributive and $\xi \circ \mu X = \xi \circ \tilde{G}\xi$, the operations $\wedge$ and $\vee$ are such ones. By the construction, $\xi(\mathcal{A}) = \vee\{\wedge A \mid A \in \mathcal{A}\}$ for all $\mathcal{A} \in \tilde{G}X$.

Sufficiency. Let X be a compact with the considered structures. Define a map $\tilde{G}\colon X \to X$ by $\xi(\mathcal{A}) = \vee\{\wedge A \mid A \in \mathcal{A}\}$. Clearly, ξ is continuous. The equality $\xi \circ \eta X = \mathrm{id}_X$ is obvious. Since the operations $\wedge$ and $\vee$ are completely distributive, we have also the equality $\xi \circ \mu X = \xi \circ G\xi$. $\square$

It can be verified directly that morphisms of $\tilde{\mathbb{G}}$-algebras can be characterized as complete homomorphisms on the both operations.

Let X be an ordered compact Hausdorff space. Then X is said to be the *Lawson k-v-set* if there exists $\sup A$ for every $A \in \exp X$, every

k points of which have a sup, and moreover, the map $\sup\colon \mathcal{A} \to X$ is continuous on a maximal domain $\mathcal{A} \subset \exp X$ of existing of sup. We use the term the *Lawson binary v-set* instead of the Lawson 2-v-set. If in this definition one writes "every finite set" instead of the words "every k points", he obtains a notion of the *Lawson ∞-v-set*.

Lemma 3.6.31. *Let (X, ξ) be an $\widetilde{\mathbb{N}}$-algebra, $\xi(\mathcal{A}_1) = \xi(\mathcal{A}_2)$, $\xi(\mathcal{B}_1) = \xi(\mathcal{B}_2)$, where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \in \widetilde{N}X$ and $\mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1, \mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2 \in \widetilde{N}X$. Then*

$$\xi(\mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1 = \xi(\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2).$$

Proof. Consider the following points of $\widetilde{N}^2 X$:

$$\mathfrak{A}_1 = \{\{\mathcal{A}_1, \mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2\}, \{\mathcal{B}_1, \mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2\}, \{\mathcal{A}_1, \mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2, \mathcal{B}_1\}\}.$$

Then

$$\mu X(\mathfrak{A}_1) = (\mathcal{A}_1 \widetilde{\cap} (\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2)) \widetilde{\cup} (\mathcal{B}_1 \widetilde{\cap} (\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2)) = (\mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1) \widetilde{\cap} (\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2)$$

and

$$\begin{aligned}\widetilde{N}\xi(\mathfrak{A}_1) &= \gamma X(\{\{\xi(\mathcal{A}_1), \xi(\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2)\}, \{\xi(\mathcal{B}_1), \xi(\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2)\}\}) = \\ &= \gamma X(\{\{\xi(\mathcal{A}_2), \xi(\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2)\}, \{\xi(\mathcal{B}_2), \xi(\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2)\}\}) = \\ &= \widetilde{N}\xi(\mathfrak{A}_2),\end{aligned}$$

where

$$\mathfrak{A}_2 = \gamma \widetilde{N} X(\{\{\mathcal{A}_2, \mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2\}, \{\mathcal{B}_2, \mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2\}\}),$$

and the retraction $\gamma X\colon \exp^2 X \to \widetilde{G}X$ is defined in subsection 2.1.4. Besides,

$$\mu X(\mathfrak{A}_1) = (\mathcal{A}_2 \widetilde{\cap} (\mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1)) \widetilde{\cup} (\mathcal{B}_2 \widetilde{\cap} (\mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1)) = \widetilde{N}\xi(\mathfrak{A}_3),$$

where

$$\mathfrak{A}_3 = \gamma \widetilde{N} X(\{\{\mathcal{A}_1, \mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1\}, \{\mathcal{B}_1, \mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1\}\}).$$

Moreover, we have $\mu X(\mathfrak{A}_2) = \mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2$ and $\mu X(\mathfrak{A}_3) = \mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1$. Hence,

$$\xi(\mathcal{A}_1 \widetilde{\cup} \mathcal{B}_1) = \xi(\mathcal{A}_2 \widetilde{\cup} \mathcal{B}_2),$$

as required. □

Now we obtain a characterization of the $\widetilde{N}$-algebras.

Theorem 3.6.32. *A couple (X, ξ) is an $\widetilde{\mathbb{N}}$-algebra iff there exist jointly completely distributive structures of the Lawson semilattices (inf) and the Lawson binary v-set ($\sup_1$) on X, satisfying $\xi(\mathcal{A}) = \sup_1\{\inf A \mid A \in \mathcal{A}\}$, $\mathcal{A} \in \widetilde{N}X$.*

Proof. Let X be an $\widetilde{\mathbb{N}}$-algebra. Let the Lawson semilattice structure be constructed as in Theorem 3.6.30. Form the Lawson binary v-set structure on X in the following manner. Consider the order on X: $x \leq y$ iff there exist $\mathcal{A}_1 \in \xi^{-1}(x)$, $\mathcal{A}_2 \in \xi^{-1}(y)$ with $\xi(\mathcal{A}_1 \widetilde{\cup} \mathcal{A}_2) = y$. By Lemma 3.6.31, this order is well-defined. Let $A \in \exp X$ and there exist $\sup\{a, b\}$ for all $a, b \in A$. Now construct $\sup A$. For every $a \in A$ put $a^+ = \{b \mid a \leq b\} \in \exp X$. Since there exists sup for every two points of A, the set $\mathcal{A} = \gamma X\{a^+ \mid a \in A\}$ is linked. It is easy to see that $\xi(\mathcal{A}) = \sup_1 A$.

Show continuity of $\sup_1$. For this we have only to prove that the map $+ \colon X \to \exp X$, $+(a) = a^+$, is continuous.

Let $a^+ \in \langle V \rangle$ for an open set $V \subset X$. Suppose that there exists a sequence $\{a_i\}$ converging to a and such that for every i the set a^+ contains a point b_i with $b_i \notin V$. Since X and $\widetilde{N}X$ are compact, we can choose a subsequence $\{a_{i_k}, b_{i_k}\}$, satisfying the following conditions:

1) $b_{i_k} \to b$;
2) there exist elements $\mathcal{A}_{i_k} \in \xi^{-1}(a_{i_k})$ and $\mathcal{B}_{i_k} \in \xi^{-1}(b_{i_k})$ such that $\xi(\mathcal{A}_{i_k} \widetilde{\cup} \mathcal{B}_{i_k}) = b_{i_k}$ and $\{\mathcal{A}_{i_k}, \mathcal{B}_{i_k}\} \to \{\mathcal{A}, \mathcal{B}\}$.

Then $\xi(\mathcal{A}) = a$, $\xi(\mathcal{B}) = b$ and $\xi(\mathcal{A} \widetilde{\cup} \mathcal{B}) = b$. Hence, $b \in a^+ \subset V$, and we obtain a contradiction. Thus, the map $+$ is continuous, and consequently, X is the Lawson binary v-set. One can show the joint complete distributivity of the constructed structures in a similar way as in Theorem 3.6.30.

Sufficiency can be also obtained similarly as in Theorem 3.6.30. Note only that existence of $\sup_1$ in the structure map follows from the fact that the considered systems are linked. □

The proof of the following results is left to the reader.

Theorem 3.6.33. *A couple (X, ξ) is an $\widetilde{\mathbb{N}}_k$-algebra iff there exist jointly comletely distributive Lawson semilattice and Lawson k-v-set structures on X, satisfying the condition*

$$\xi(\mathcal{A}) = \sup_1\{\inf A \mid A \in \mathcal{A}\}, \quad \mathcal{A} \in \widetilde{N}_k X.$$

Theorem 3.6.34. *A couple (X,ξ) is an $\widetilde{\mathbb{N}}_\infty$-algebra iff there exist jointly comletely distributive Lawson semilattice and Lawson ∞-v-set structures on X, satisfying the condition*

$$\xi(\mathcal{A}) = \sup_1\{\inf A \mid A \in \mathcal{A}\}, \quad \mathcal{A} \in \widetilde{N}_\infty X.$$

Finally, remark the obvious fact that morphisms of $\widetilde{\mathbb{N}}$-, $\widetilde{\mathbb{N}}_k$-, $\widetilde{\mathbb{N}}_\infty$-algebras are characterized by complete homomorphisms of the respective structures.

3.7. Lifting of functors to the category of $\mathbb{T}$-algebras

Proposition 3.7.1. *There exists a lifting of the functor T to the Eilenberg-Moore category $\mathcal{C}^{\mathbb{T}}$ of every monad $\mathbb{T} = (T,\eta,\mu)$.*

Proof. Put $\delta = T\eta \circ \mu$ and appeal to Theorem 1.2.6. $\square$

Proposition 3.7.2. *Let $\mathbb{T}$ be a projective monad. Then every endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ has a lifting to the category $\mathcal{C}^{\mathbb{T}}$.*

Proof. Let $\pi\colon T \to 1_{\mathcal{C}}$ be the projection. It is easy to check that the natural transformation $\delta = F\eta \circ \pi F\colon TF \to FT$ satisfies the conditions 1) and 2) of Theorem 1.2.6. $\square$

Proposition 3.7.3. *Let $\mathcal{C}$ be a category with products. Then the power functor $(-)^\alpha\colon \mathcal{C} \to \mathcal{C}$ has a lifting to the category $\mathcal{C}^{\mathbb{T}}$.*

Proof. Define $\delta\colon T(-)^\alpha \to (-)^\alpha T$ by the conditions $\mathrm{pr}_i \circ \delta X = T\,\mathrm{pr}_i$, $i \in \alpha$, and use Theorem 1.2.6. $\square$

Now we need some notations.

Let (F_1, F_2) be a couple of normal functors, and $a \in F_2X_1$, $b \in F_1X_2$. For each $y \in X_2$, let $i_y\colon X_1 \to X_1 \times X_2$ be a map sending $x \in X_1$ to $(x,y) \in X_1 \times X_2$. It is easy to verify that the map $y \mapsto i_y$ is a continuous map of X_2 to $C(X_1, X_1 \times X_2)$. Using this and continuity of F_2, obtain that the following map is continuous: $f_a\colon X_2 \to F_2(X_1 \times X_2)$, $f_a(y) = F_2 i_y(a)$. Put

$$a \oplus b = F_1 f_a(b) \in F_1F_2(X_1 \times X_2).$$

Similarly, define the map $j_x\colon X_2 \to X_1 \times X_2$ by $j_x(y) = (x, y)$, $y \in X_2$, and the map $g_b\colon X_1 \to F_1(X_1 \times X_2)$ by $g_b(x) = F_1 j_x(b)$, $x \in X_1$. Put

$$a \mathbin{\widetilde{\oplus}} b = F_2 g_b(a) \in F_2 F_1(X_1 \times X_2).$$

We have immediately

Lemma 3.7.4. *The operations $\oplus$ and $\widetilde{\oplus}$ are natural by both arguments.*

Let $\deg F = n < \omega$ and $a \in Fn$, $\deg(a) = n$. Let

$$(F/a)X = \{b \in F(n \times X) \mid F\,\mathrm{pr}_1(b) = a\}.$$

It is easy to see that F/a is a subfunctor of $F(n\times(-))$ and is isomorphic to the power functor $(-)^n$. For each $m \in n$, define the natural transformations $\pi_m\colon F/a \to 1_{\mathbf{Comp}}$ by the property: $\pi_m X(b) = y$, where $(m, y \in \mathrm{supp}_F(b)$.

Fiven normal functors F, F' and $a \in FF'X$ let

$$\mathrm{Supp}(a) = \{\mathrm{supp}\, F'(b) \mid b \in \mathrm{supp}_F(a)\}.$$

The following theorem shows that in the case of a normal monad $\mathbb{T}$ all the situations when there exists a lifting of a functor of finite degree to the category $\mathbf{Comp}^{\mathbb{T}}$ are described by Propositions 3.7.2 and 3.7.3.

Theorem 3.7.5. *Let a normal functor F of finite degree $n \geq 1$ has a lifting to the category $\mathbf{Comp}^{\mathbb{T}}$ of $\mathbb{T}$-algebras of a normal monad $\mathbb{T} = (T, \eta, \mu)$. Then either $F \cong (-)^n$ or $\mathbb{T}$ is a projective monad.*

Proof. Here we use the denotation $\oplus$ for the case $F_1 = T$, $F_2 = F$. For each X define the map $kX\colon n \times TX \to T(n \times X)$ by the formula

$$kX(m, b) = Tj_m(b), \quad m \in n, \quad b \in TX.$$

Suppose that F has a lifting to the category $\mathbf{Comp}^{\mathbb{T}}$ and let $\delta\colon TF \to FT$ be the natural transformation corresponding to this lifting by Theorem 1.2.6. Then for each $b \in TX$ we have

$$FT\mathrm{pr}_1 \circ \delta(n \times X)(a \oplus b) = \delta X \circ TF\,\mathrm{pr}_1(a \oplus b) = \delta X \circ \eta Fn(a) = F\eta n(a).$$

Fix $a \in Fn$ with $\deg_F(a) = n$. Then

$$\delta(n \times X)(a \oplus b) \in FkX(n \times TX)$$

and

$$FkX^{-1} \circ \delta(n \times X)(a \oplus b) \in (F/a)TX.$$

Hence, for each $m \in n$ we can well define a natural transformation $\psi_m : T \to T$ by the formula:

$$\psi_m X(b) = \pi_m TX \circ FkX^{-1} \circ \delta(n \times X)(a \oplus b), \ b \in TX.$$

Lemma 3.7.6. *For each $d \in T(F/a)X \subset TF(n \times X)$ we have $\pi_m TX \circ FkX^{-1} \circ \delta(n \times X)(d) = \psi_m X \circ T\pi_m X(d)$.*

Proof. For some X' and a map $\alpha : n \times X \to n \times X$ there exists $b \in TX'$ such that $TF\alpha(a \oplus b) = d$. For each $l \in n$ there exists a map $\alpha_l : X' \to X$ such that $\alpha(l, x) = (l, \alpha_l(x))$, $x \in X$. Note that $T\alpha_l(b) = T\pi_l X(d)$. We have

$$\begin{aligned}
&\pi_m TX \circ FkX^{-1} \circ \delta(n \times X)(d) = \\
&\quad = \pi_m TX \circ FkX^{-1} \circ \delta(n \times X) \circ TF\alpha(a \oplus b) = \\
&\quad = \pi_m TX \circ FkX^{-1} \circ FT\alpha \circ \delta(n \times X')(a \oplus b) = \\
&\quad = T\alpha_m \circ \pi_m TX \circ FkX'^{-1} \circ \delta(n \times X')(a \oplus b) = \\
&\quad = T\alpha_m \circ \psi_m X'(b) = \psi_m X \circ T\alpha_m(b) = \psi_m X \circ T\pi_m X(d).
\end{aligned}$$

□

Lemma 3.7.7. *The natural transformation $\psi_m : T \to T$ is a morphism of the monad $\mathbb{T}$ into itself.*

Proof. Since components of a natural transformation of normal functors do not enlarge supports, we have $\psi_m X \circ \eta X(x) = \eta X(x)$, $x \in X$.

If $\mathcal{B} \in T^2X$, then, by Lemma 3.7.6,

$$\begin{aligned}
&\psi_m X \circ \mu X(\mathcal{B}) \\
=&\pi_m TX \circ FkX^{-1} \circ \delta(n \times X) \circ Tf_a \circ \mu X(\mathcal{B}) \\
=&\pi_m TX \circ FkX^{-1} \circ F\mu(n \times X) \circ \delta T(n \times X) \circ T\delta(n \times X) \circ T^2 f_a(\mathcal{B}) \\
=&\pi_m TX \circ F(1_n \times \mu X) \circ FkTX^{-1} \circ \delta T(n \times X) \circ T\delta(n \times X) \circ T^2 f_a(\mathcal{B}) \\
=&\mu X \circ \pi_m T^2 X \circ FkX^{-1} \circ \delta(n \times TX) \\
&\quad \circ T(FkX^{-1} \circ \delta(n \times X) \circ Tf_a)(\mathcal{B}) \\
=&\ \mu X \circ \pi_m T^2 X \circ FkTX^{-1} \circ \delta(n \times TX) \\
&\quad \circ T(a \oplus T\pi_m TX \circ T(FkX^{-1} \circ \delta(n \times X) \circ Tf_a)(\mathcal{B})) \\
=&\mu X \circ \pi_m T^2 X \circ FkTX^{-1} \circ \delta(n \times TX) \circ T(a \oplus T\psi_m X(\mathcal{B})) \\
=&\mu X \circ \psi_m TX \circ T\psi_m X(\mathcal{B}).
\end{aligned}$$

□

Return to the proof of Theorem 3.7.5. Every subfunctor $T_m = \psi_m(T)$ of T generates, by Lemma 3.7.7, the submonad $\mathbb{T}_m = (T_m, \eta, \mu|T_m^2)$ of $\mathbb{T}$, $m \in n$. Suppose that $\mathbb{T}$ is not projective, then $\mathbb{T}_m$ is also such a one, and by the result of Theorem 3.4.4, $\deg(T_m) = \infty$.

From elementary properties of normal functors, we can easily deduce that there exist a discrete two-point space $X = \{x_1, x_2\}$ and $b \in TX$ such that

$$\mathrm{Supp}_{FT}(\delta(n \times X)(a \oplus b)) = \{\{m\} \times X \mid m \in n\}.$$

Show that for any maps $f_1, f_2 \colon n \to n$ such that $f_1 \neq f_2$, we have $Ff_1(a) \neq Ff_2(a)$. Assuming the contrary, define the maps $h_1, h_2 \colon n \times X \to n \times X$ by

$$h_1 = f_1 \times 1_X, \ h_2(m, x_i) = (f_i(m), x_i), \ m \in n, \ i = 1, 2.$$

Then obviously $TFh_1(a \oplus b) = TFh_2(a \oplus b)$, but

$$\begin{aligned}
&\mathrm{Supp}_{FT}(\delta(n \times X) \circ TFh_1(a \oplus b)) = \{h_1(\{m\} \times X) \mid m \in n\} \neq \\
&\quad \neq \{h_2(\{m\} \times X) \mid m \in n\} = \mathrm{Supp}_{FT}(\delta(n \times X) \circ TFh_2(a \oplus b)),
\end{aligned}$$

and we get a contradiction.

In order to prove the isomorphism $F \cong (-)^n$, by Theorem 2.9.7, it is sufficient to prove that for each $a' \in Fn$ there exists a map $f: n \to n$ with $Ff(a) = a'$. Define the map $l: X \to F(n \times X)$ by $l(x_1) = a'$, $l(x_2) = a$, and let $d = Tl(b)$.

For $i = 1, 2$ denote by $\mathcal{R}_i$ the partition of $n \times X$, whose a unique non-trivial element is the set $n \times \{x_i\}$. Let $Y_i = (n \times X)/\mathcal{R}_i$ be a quotient space and $q_i: n \times X \to Y_i$ be a quotient map, $i = 1, 2$. Then obviously

$$TFq_1(d) = TFq_1(a \oplus b),$$

and consequently, for each $A \in \operatorname{Supp}_{FT}\big(FTq_1 \circ \delta(n \times X)(d)\big)$ the set $A \cap (n \times \{x_2\})$ is a singleton. Similary, for each $B \in \operatorname{Supp}_{FT}\big(FTq_2 \circ \delta(n \times X)(d)\big)$, the set $B \cap (n \times \{x_1\})$ is a singleton.

Finnaly, we obtain that for each $C \in \operatorname{Supp}_{FT}\big(\delta(n \times X)(d)\big)$ and for each $x \in X$, the set $C \cap (n \times \{x\})$ is a singleton. The intersections $C \cap (n \times \{x_2\})$, where $C \in \operatorname{Supp}_{FT}\big(\delta(n \times X)(d)\big)$, form a disjoint cover of the set $n \times \{x_2\}$.

Now we can construct the desired map f by the following manner. Let $m \in n$. There exists a (unique as remarked above) element $C_m \in \operatorname{Supp}_{FT}\big(\delta(n \times X)(d)\big)$ such that $(m, x_2) \in C_m$. Take $f(m) \in n$ for which $\big(f(m), x_1\big) \in C_m$. □

Exercises

1. Let $\mathbb{T} = (T, \eta, \mu)$ be a (weakly, almost) normal monad in **Comp**. For $X \in |\mathbf{Comp}|$ let $A_{\mathbb{T}}X = \{C \in \exp TX \mid (C, \mu X|TC$ is a subalgebra in $(TX, \mu X)\}$.
   (a) Prove that $A_{\mathbb{T}}X$ is a closed subset in $\exp TX$.
   (b) Prove that $A_{\mathbb{T}}$ is a subfunctor of $\exp T$. Is $A_{\mathbb{T}}$ (weakly, almost) normal?
2. Let $\mathbb{T} = (T, \eta, \mu)$ be a (weakly, almost) normal monad in **Comp**. A nonempty closed subset $\mathcal{A}$ of $A_{\mathbb{T}}X$ is called a growth hyperspace in $A_{\mathbb{T}}X$ if $C \subset D$, $C \in \mathcal{A}$, $D \in A_{\mathbb{T}}X$ implies $D \in \mathcal{A}$.
   (a) Prove that the set $G_{\mathbb{T}}X$ of all growth hyperspaces is closed in $\exp^2 TX$.
   (b) For a map $f: X \to Y$ and $\mathcal{A} \in G_{\mathbb{T}}X$ let $G_{\mathbb{T}}f(\mathcal{A})$ be a minimal growth hyperspace in $A_{\mathbb{T}}Y$ containing the set $\{A_{\mathbb{T}}f(C) \mid C \in A_{\mathbb{T}}X\}$. For what $\mathbb{T}$ the functor $G_{\mathbb{T}}$ is an almost normal functor in **Comp**?

3.8. Distributive law for normal monads

If $\mathbb{T}_1 = (T_1, \eta_1, \mu_1)$ and $\mathbb{T}_2 = (T_2, \eta_2, \mu_2)$ are monads on a category, it is natural to ask whether the composition T_2T_1 produces a monad. We answer to this question in the case of (weakly, almost) normal monads.

Definition 3.8.1. A *distributive law* of a monad $\mathbb{T}_1 = (T_1, \eta_1, \mu_1)$ over a monad $\mathbb{T}_2 = (T_2, \eta_2, \mu_2)$ is a natural transformation $\lambda\colon T_1T_2 \to T_2T_1$ satisfying the relations:

(D1) $\lambda \circ T_1\eta_2 = \eta_2 T_1$;
(D2) $\lambda \circ \eta_1 T_2 = T_2\eta_1$;
(D3) $\lambda \circ T_1\mu_2 = \mu_2 T_1 \circ T_2\lambda \circ \lambda T_2$;
(D4) $\lambda \circ \mu_1 T_2 = T_2\mu_1 \circ \lambda T_1 \circ T_1\lambda$.

Examples. Suppose that T_1 is a (weakly, almost) normal functor and $T_1 = (-)^\alpha$, $1 \le \alpha \le \omega$. Define $\lambda = (\lambda X)$ by the formula $\lambda X(a) = (T_1 \mathrm{pr}_i X(a))_{i<\alpha}$ ($\mathrm{pr}_i\colon (-)^\alpha \to \mathrm{Id}$ is the i-th projection natural transformation).

Proposition 3.8.2. *Let $\lambda\colon T_1T_2 \to T_2T_1$ be a distributive law of a monad $\mathbb{T}_1 = (T_1, \eta_1, \mu_1)$ over a monad $\mathbb{T}_2 = (T_2, \eta_2, \mu_2)$. Then the triple (T_2T_1, η, μ), where $\eta = \eta_2T_1 \circ \eta_1$, $\mu = \mu_2T_1 \circ T_2^2\mu_1 \circ T_2\lambda T_1$ is a monad.*

Proof. Straightforward. □

The main result of this section is the following

Theorem 3.8.3. *Let $\mathbb{T}_i = (T_i, \eta_i, \mu_i)$, $i = 1, 2$, be normal monads in the category* **Comp**, *$\mathbb{T}_i \neq (\mathrm{Id}, \mathrm{id}, \mathrm{id})$, and there exist a distributive law of the monad $\mathbb{T}_1$ over the monad $\mathbb{T}_2$. Then $\mathbb{T}_2$ is a power monad.*

Proof. Here we use the notations $\oplus$ and $\widetilde{\oplus}$ for the couple (T_1, T_2) (see the previous section). For each spaces X and Y and each $a \in T_2X$, $b \in T_1Y$, we obtain

$$T_1T_2\,\mathrm{pr}_1(a \oplus b) = \eta_1T_2X(a), \quad T_1T_2\,\mathrm{pr}_2(a \oplus b) = T_1\eta_2Y(b).$$

Denoting the distributive law by λ and using the properties (D1), (D2), we obtain

$$T_2T_1\mathrm{pr}_1 \circ \lambda(X \times Y)(a \oplus b) = T_2\eta_1X(a),$$
$$T_2T_1\mathrm{pr}_2 \circ \lambda(X \times Y)(a \oplus b) = \eta_2T_1Y(b).$$

Hence, $\lambda(X \times Y)(a \oplus b) = a \widetilde{\oplus} b$.

Show that T_2 is a profinitely power functor. Let X be finite discrete, $\mathrm{supp}_{T_2}(a) = X$, and $b \in T_1Y$ satisfy $\deg(b) = 2$. Choose $y_1, y_2 \in \mathrm{supp}_{T_1}(b)$, $y_1 \neq y_2$. It is sufficient to show that for all maps $f_1, f_2\colon X \to$

X, we have $T_2 f_1(a) \neq T_2 f_2(a)$, whenever $f_1 \neq f_2$. Assuming the contrary, define the maps $h_1, h_2 \colon X \times Y \to X \times Y$ by

$$h_1 = f_1 \times \mathrm{id}_y,$$

$$h_2(x, y) = \begin{cases} (f_1(x), y), & \text{if } y \neq y_2; \\ (f_2(x), y), & \text{otherwise.} \end{cases}$$

Then, by the assumption, $T_1 T_2 h_1(a \oplus b) = T_1 T_2 h_2(a \oplus b)$. Similarly to the proof of Theorem 3.7.5 we can verify that

$$\lambda(X \times Y) \circ T_1 T_2 h_1(a \oplus b) \neq \lambda(X \times Y) \circ T_1 T_2 h_2(a \oplus b),$$

thus obtaining a contradiction.

As a matter of fact, it is proved by Theorem 3.4.4 that every normal functor determining a monad, is weakly bicommutative. Now, it follows from Theorem 2.9.7 that T_2 is a power functor. □

There is no counterpart of Theorem 3.8.3 for the monads in the category $\mathbf{Comp}^\infty$. Let (FSG, η_1, μ_1) and (AG, η_2, μ_2) be the monads of free topological semigroup and free abelian topological group respectively. Define a distributive law $FSG\ AG \to AG\ FSG$ by the formula (we use additive notation for the elements of the free abelian topological group):

$$\begin{aligned} &\lambda X((\zeta_{11} x_{11} + \dots + \zeta_{1k_1} x_{1k_1})^{\varepsilon_1} \dots (\zeta_{n1} x_{n1} + \dots + \zeta_{nk_n} x_{nk_n})^{\varepsilon_n} \\ =&(\zeta_{11}^{\varepsilon_1} \dots (\zeta_{n1}^{\varepsilon_n}) x_{11}^{\varepsilon_1} \dots x_{n1}^{\varepsilon_n} \dots \end{aligned}$$

(here $\zeta_{ij} = \pm 1$).

Then the composition of the above monads is the free topological ring monad.

3.9. Perfectly metrizable monads

Definition 3.9.1. A (weakly, almost) normal functor F in **Comp** is called *metrizable* if there exists a correspondence which assigns to each metric d_X on a compact metrizable space X a metric d_{FX} on FX so that the following conditions hold:

1) for every isometric embedding $i \colon (X, d) \to (X', d')$ the map

$$Fi \colon (FX, d_{FX}) \to (FX', d'_{FX})$$

is also an isometric embedding;

2) the natural embedding $\eta X\colon (X,d) \to (FX, d_{FX})$ is an isometric embedding;
3) $\operatorname{diam}(FX, d_{FX}) = \operatorname{diam}(X,d)$.

The above correspondence $d_X \mapsto d_{FX}$ is called a *metrization* of the functor F. For $n \le m$ let

$$\eta_{n,m}X = \eta F^{m-1}X \circ \cdots \circ \eta F^n X.$$

Fix a metrization of the functor F. Then any metric d on a compact metrizable space X generates a metric d_{F^nX} on F^nX. Simplifying the notation, we denote the latter metric by d_n. Consider the direct sequence

$$(X,d) \xrightarrow{\eta_{0,1}X} (FX, d_1) \xrightarrow{\eta_{1,2}X} (F^2X, d_2) \longrightarrow \cdots$$

and denote its limit (in the category of metric spaces and isometric embeddings) by (F^+X, d_X^+). Let $\eta_n\colon F^nX \to F^+X$ be the natural inclusions. Every map $f\colon X \to Y$ of compact metric spaces obviously determines the natural map $F^+f\colon F^+X \to F^+Y$. By $\eta_{n,+}$ we denote the natural embedding of F^n into F^+.

Definition 3.9.2. A metrization of F is called *uniformly continuous* if for every map $f\colon X \to Y$ of compact metric spaces the induced map $F^+f\colon F^+X \to F^+Y$ is uniformly continuous.

From now on, suppose that the metrization of F under consideration is uniformly continuous. Then for every homeomorphism $f\colon X \to Y$ of compact metric spaces the map $F^+f\colon F^+X \to F^+Y$ is a uniform homeomorphism. Thus, the induced uniform structure (and topology) on F^+X fails to depend on a concrete metric on X, and we can consider F^+ as a functor from the category $M\mathbf{Comp}$ of compact metrizable spaces into the category $\mathbf{Unif}$ uniform spaces and uniformly continuous maps. Denoting by $C\colon \mathbf{Unif} \to \mathbf{Unif}$ the completion functor, let $F^{++} = CF^+$. Now, suppose that F generates a monad $\mathbb{F} = (F, \eta, \mu)$.

Definition 3.9.3. The monad $\mathbb{F}$ is called *perfectly metrizable* if the map $\mu X\colon (F^2X, d_2) \to (FX, d_1)$ is nonexpanding.

We will use the following denotation: for $1 \leq n < m$ let

$$\mu_{m,n} = \mu F^n X \circ \cdots \circ \mu F^{m-2} X \circ \mu F^{m-1} X.$$

Consider the inverse sequence

$$FX \xleftarrow{\mu_{2,1} X} F^2 X \xleftarrow{\mu_{3,2} X} F^3 X \longleftarrow \cdots$$

and let $(F^\omega X, \mu_{\omega,i} X)$ denote its limit. For every $n \geq 1$ let us denote by $\vartheta_n X \colon F^+ \to F^n X$ the map defined by the conditions $\vartheta_n X \circ \eta_m X = \mu_{m,n} X$, $m \in \mathbb{N}$. By definition 3.9.3, the map $\vartheta_n X$ is nonexpanding. Hence, it can be continuously extended to a map $\bar{\vartheta}_n X \colon F^{++} X \to F^n X$. For $m < n$ we have $\vartheta_m X = \mu_{n,m} X \circ \vartheta_n X$ and, therefore, $\bar{\vartheta} X = \mu_{n,m} X \circ \bar{\vartheta} X$. Thus, there exist the limit maps

$$\vartheta X \colon F^+ X \to F^\omega X, \quad \bar{\vartheta} X \colon F^{++} X \to F^\omega X,$$

i.e., the maps satisfying the conditions

$$\mu_n X \circ \vartheta X = \vartheta_n X, \quad \mu_n X \circ \bar{\vartheta} X = \bar{\vartheta}_n X, \qquad n \in \mathbb{N}.$$

Proposition 3.9.4. *The maps*

$$\eta_n X \circ \bar{\vartheta}_n X \colon F^{++} X \to F^n X \to F^+ \hookrightarrow F^{++} X, \qquad n \in \mathbb{N},$$

converge to the identity uniformly on the compact subsets.

Proof. For every $x \in FX^{++}$ and $x' \in \eta_n X(F^n X) \subset F^{++} X$ we have

$$\begin{aligned}
d^{++}(\eta_n X \circ \bar{\vartheta}_n X(x), x) &\leq d^{++}(x, x') + d(x', \eta_n X \circ \bar{\vartheta}_n X(x)) \\
&= d^{++}(x, x') + d(\eta_n X \circ \bar{\vartheta}_n X(x'), \eta_n X \circ \bar{\vartheta}_n X(x)) \\
&\leq 2d^{++}(x, x').
\end{aligned}$$

Thus,

$$d^{++}(\eta_n X \circ \bar{\vartheta}_n X(x), x) \leq 2d^{++}(x, \eta_n X(F^n X)).$$

This implies that for any $y \in F^{++} X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
&d^{++}(x, \eta_n X \circ \bar{\vartheta}_n X(y)) \\
&\quad \leq d^{++}(x, \eta_n X \circ \bar{\vartheta}_n X(x)) + d^{++}(\eta_n X \circ \bar{\vartheta}_n X(x), \eta_n X \circ \bar{\vartheta}_n X(y)) \\
&\quad \leq 2d^{++}(x, \eta_n X(F^n X)) + d^{++}(x, y).
\end{aligned}$$

Now, let $K \subset F^{++}X$ be a compact subset. For every $\varepsilon > 0$ there exists a finite ε-net $k_1, \dots, k_m$ in K. There exists $n \in \mathbb{N}$ such that if $j \geq n$, then $d^{++}(k_i, \eta_j X(F^j X)) < \varepsilon$, for every $i = 1, \dots, m$. For $x \in K$ and $j \geq n$ we have

$$\begin{aligned} d^{++}(\eta_j X \circ \bar{\vartheta}_j X(x), x) &\leq d^{++}(\eta_j X \circ \bar{\vartheta}_j X(x), \eta_j X \circ \bar{\vartheta}_j X(k_i)) + \\ &+ d^{++}(\eta_j X \circ \bar{\vartheta}_j X(k_i), k_i) + d^{++}(k_i, x) \leq \\ &\leq 2d^{++}(x, k_i) + 2d^{++}(k_i, \eta_j X(F^j X)) < 4\varepsilon. \quad \square \end{aligned}$$

Theorem 3.9.5. *For every X and a perfectly metrizable F the map $\bar{\vartheta}X : F^{++}X \to F^{\omega}X$ is an embedding.*

Proof. Proposition 3.9.4 implies injectivity of $\bar{\vartheta}X$. We are going to show that the map $(\bar{\vartheta}X)^{-1} \colon \bar{\vartheta}X(F^{++}X) \to F^{++}X$ is continuous. For every $x \in F^{++}X$ and $\varepsilon > 0$ it is necessary to find a neighborhood U of $\bar{\vartheta}X(x)$ in $F^{\omega}X$ such that

$$\bar{\vartheta}^{-1}(U) \subset O_{\varepsilon}(x). \tag{3.18}$$

By Proposition 3.9.4 the sequence $\eta_n X \circ \bar{\vartheta}_n X(x)$ converges to x. Therefore, there exists a natural n such that

$$d_{++}(\eta_k X \circ \bar{\vartheta}_k X(x), x) < \frac{\varepsilon}{4} \text{ for every } k \geq n. \tag{3.19}$$

Put $V = O_{\frac{\varepsilon}{2}}(\eta_{n,n+1} X \circ \bar{\vartheta}_n X(x)) \subset F^{n+1}X$ and $U = \psi_{n+1}^{-1} X(V)$.

Let us verify the inclusion

$$F^{+}X \cap (\vartheta X)^{-1}(U) \subset O_{\frac{3\varepsilon}{4}}(x). \tag{3.20}$$

Indeed, suppose that $y \in F^{+}X \cap (\vartheta X)^{-1}(U)$. By the definition of $F^{+}X$ we see that there exists $k \geq n+2$ such that $y = \eta_k X(z)$, where $z \in F^k X$. Since $y \in (\vartheta X)^{-1}(U)$, we have $\psi_{k,n+1} X(z) \in V$. Consequently, using Proposition 3.9.4 and the fact that $d_{l+1}(\psi_{m,l+1} X(a), \eta_{l,l+1} X(x')) = d_m(a, \eta_{n,m} X(x')$ for every $a \in F^m X$, $x' \in F^l X$, $m \geq l+2$, we obtain

$$\begin{aligned} \frac{\varepsilon}{2} > d_{n+1}(\psi_{k,n+1} X(z), \eta_{n,n+1} X \circ \bar{\vartheta}_n X(x)) &= d_k(z, \eta_{n,k} X \circ \bar{\vartheta}_n X(x)) = \\ = d_{+}(\eta_k X(z), \eta_k X \circ \eta_{n,k} X \circ \bar{\vartheta}_n(x)) &= d_{+}(y, \eta_n X \circ \bar{\vartheta}_n X(x)). \end{aligned}$$

Then

$$d_{++}(y,x) \leq d_+(y, \eta_n X \circ \bar{\vartheta}_n X(x)) + d_{++}(\eta_n X \circ \bar{\vartheta}_n X(x), x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}$$

(here the inequality $<$ is a consequence of (3.19)). Thus, the inclusion (3.20) is verified. Since F^+X is dense in $F^{++}X$, the inclusion (3.20) implies

$$\theta^{-1}(U) \subset \overline{O_{\frac{3\varepsilon}{4}}(x)} \subset O_\varepsilon(x).$$

Therefore, we have (3.18). □

We may consider F^+ and F^{++} as functors from M**Comp** to **Tych**.

Theorem 3.9.6. *The functor F^ω : M**Comp** $\to$ **Comp** is a compactification of the functors F^+, F^{++} : M**Comp** $\to$ **Tych**.*

Proof. Since the maps $\mu_n X \circ \vartheta X$ map F^+X onto $F^n X$, we see that $F+X$ and therefore $F^{++}X$ are dense in $F^\omega X$. The result now follows from Theorem 3.9.5 and naturality of $\bar{\vartheta}$. □

Proposition 3.9.7. *Suppose that $\mathbb{F} = (F, \eta, \mu)$ is a perfectly metrizable monad. Then F^+ is a right $\mathbb{F}$-functor. If additionally the metrization of F satisfies the following condition:*

(*) for every nonexpanding map $f\colon (X_1, d_{X_1}) \to (X_2, d_{X_2})$ the map $Ff\colon (FX_1, d_{FX_1}) \to (FX_2, d_{FX_2})$ is nonexpanding,

then F^{++} is also a right $\mathbb{F}$-functor.

Proof. First, show that F^+ is a right $\mathbb{F}$-functor. Define the natural transformation $\lambda_+\colon F^+F \to F^+$ by the following data:

$$\lambda_+ \circ \eta_{n,+}F = \eta_{n,+} \circ F^{n-1}\mu, \ n \in \mathbb{N};$$

λ_+ is well-defined, because of the equality

$$\lambda_+ \circ \eta_{n,+}F \circ \eta_{n,n+1}F = \eta_{n,+} \circ F^{n-1}\mu \circ \eta F^n = \eta_{n-1,+} \circ F^{n-2}\mu, n \geq 2.$$

Now,

$$\lambda_+ \circ F^+\eta \circ \eta_{n,+} = \lambda_+ \circ \eta_{n,+}F \circ F^n\eta = \eta_{n,+} \circ F^{n-1}\mu \circ F^n\eta = \eta_{n,+}, \ n \in \mathbb{N},$$

and therefore $\lambda_+ \circ F^+\eta = 1_{F^+}$.

Besides,

$$
\begin{aligned}
\lambda_+ \circ F^+\mu \circ \eta_{n,+}F^2 &= \lambda_+ \circ \eta_{n,+}F \circ F^n\mu = \eta_{n,+} \circ F^{n-1}\mu \circ F^n\mu \\
&= \eta_{n,+} \circ F^{n-1}(\mu \circ F\mu) = \eta_{n,+} \circ F^{n-1}\mu \circ F^{n-1}\mu F \\
&= \lambda_+ \circ \eta_{n,+}F \circ F^{n-1}\mu F = \lambda_+ \circ \lambda_+ F \circ \eta_{n,+}F^2, \ n \in \mathbb{N},
\end{aligned}
$$

and hence $\lambda_+ \circ F^+\mu = \lambda_+ \circ \lambda_+ F$. This means that F^+ is a right $\mathbb{F}$-functor.

If condition $(*)$, then the maps $\lambda_+ X$ are nonexpanding and therefore can be extended to the maps $\lambda_{++}X \colon F^{++}FX \to F^{++}X$. Obviously, $\lambda_{++} \circ F^{++}\mu = \lambda_{++} \circ \lambda_{++}F$. □

Now we consider the case of the functor $\exp^c$. It is metrizable by the Hausdorff metric d_{H}.

Proposition 3.9.8. *The functors* $\exp$ *and* $\exp_c$ *are uniformly metrizable.*

Proof. Given a compact metric space (X, d), we endow the space $\exp X$ with the Hausdorff metric $d_{\exp} = d_{\mathrm{H}}$ and its subspace $\exp^c X$ with the induced metric.

The conditions from the definition of uniformly metrizable functor are consequences of the following statement. If a map $f \colon (X, d^1) \to (Y, d^2)$ is (ε, δ)-uniformly continuous, then a map $\exp f \colon (\exp X, d^1_H) \to (\exp Y, d^2_H)$ is also (ε, δ)-uniformly continuous. □

Proposition 3.9.9. *The monads* $\mathbb{H}$ *and* $\mathbb{H}^c$ *are perfectly metrizable.*

Proof. Because of similarity, we consider only the case of the monad $\mathbb{H}$. As in the preceeding proposition, we endow the the space $\exp X$ with the Hausdorff metric d_{H}, for a compact metric space (X, d).

Show that the map $uX : (\exp^2 X, d_{\mathrm{HH}}) \to (\exp X, d_{\mathrm{H}}))$ is nonexpanding. Let $\mathcal{A}, \mathcal{B} \in \exp^2 X$ and $d_{\mathrm{HH}}(\mathcal{A}, \mathcal{B}) < \varepsilon$. If $a \in uX(\mathcal{A}) = \bigcup \mathcal{A}$, then $a \in A$ for some $A \in \mathcal{A}$. Since $d_{\mathrm{HH}}(\mathcal{A}, \mathcal{B}) < \varepsilon$, there exists $B \in \mathcal{B}$ such that $d_{\mathrm{H}}(A, B) < \varepsilon$,and consequently there exists $b \in B \in \bigcup \mathcal{B}$ such that $d(a, b) < \varepsilon$.

Similarly, for every $b \in B \in \bigcup \mathcal{B}$ there exists $a \in \bigcup \mathcal{A}$ with $d(a, b) < \varepsilon$. Therefore $d_{\mathrm{H}}(uX(\mathcal{A}), uX(\mathcal{B})) < \varepsilon$. □

Proposition 3.9.10. *The functor* G *is uniformly metrizable.*

Proof. For a compact metric space (X, d) endow the space GX with the metric d_{GX} induced by the Hausdorff metric d_{HH} on $\exp^2 X$. The conditions from the definition of uniformly metrizable functor can be easily verified. □

Proposition 3.9.11. *The inclusion hyperspace monad $\mathbb{G}$ is perfectly metrizable.*

Proof. Suppose that (X, d_X) is a compact metric space, $\mathfrak{A}, \mathfrak{B} \in G^2X$ and $d_{G^2X}(\mathfrak{A}, \mathfrak{B}) \leq \varepsilon$. Suppose that $A \in \mu X(\mathfrak{A})$, then there exists $\alpha \in \mathfrak{A}$ such that $A \in \cap\alpha$. By the definition of d_{G^2X}, there exists $\beta \in \mathfrak{B}$ such that $d_{\mathrm{HHH}}(\alpha, \beta) \leq \varepsilon$. For every $\mathcal{B} \in \beta$ there exists $C_{\mathcal{B}} \in \alpha$ such that $d_{\mathrm{HH}}(C_{\mathcal{B}}, \mathcal{B}) \leq \varepsilon$, and therefore there exists $B_{\mathcal{B}} \in \mathcal{B}$ such that $d_{\mathrm{H}}(B_{\mathcal{B}}, A) \leq \varepsilon$. There exists $B \in \exp X$ such that $d_{\mathrm{H}}(B, A) \leq \varepsilon$ and $B \supset B_{\mathcal{B}}$ for every $\mathcal{B} \in \beta$. Thus, $B \in \cap\beta \subset \mu X(\mathfrak{B})$.

Arguing similarly, we can show that for every $B \in \mu X(\mathfrak{B})$ there exists $A \in \mu X(\mathfrak{A})$ with $d_{\mathrm{H}}(A, B) \leq \varepsilon$. Thus $d_{GX}(\mu X(\mathfrak{A}), \mu X(\mathfrak{B})) \leq \varepsilon$. □

Proposition 3.9.12. *The probability measure functor P is uniformly metrizable.*

The technically complicated proof of this fact is omitted (see V. Fedorchuk [1990]). Here we only give the formulae for metrization.

Let (X, d_X) be a compact metric space. Define the function $d_{PX}: PX \times PX \to \mathbb{R}$ by the formula:

$$d_{PX}(\mu, \nu) = \inf\{\lambda(d_X) \mid \lambda \in P(X \times X),\ P\mathrm{pr}_1(\lambda) = \mu,\ P\mathrm{pr}_2(\lambda) = \nu\}.$$

Denote by $\mathrm{LIP}(X, d_X)$ the set of all Lipschitz functions on X (with respect to the metric d_X) and define the function $\hat{d}$: $PX \times PX \to \mathbb{R}$ by the formula:

$$\hat{d}(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi \in \mathrm{LIP}(X, d)\}.$$

Proposition 3.9.13. *The probability measure monad $\mathbb{P}$ is perfectly metrizable.*

3.9.1. Fractal objects generated by perfectly metrizable monads

Given a compact metric space (X, d_X), we endow $(C(Y, X)$, where $Y \in |\mathbf{Comp}|$, with the sup-metric,

$$d_X^{\mathrm{sup}}(f, g) = \sup\{d_X(f(y), g(y)) \mid y \in Y\}.$$

Definition 3.9.14. A metrizable functor F is called *regular* if for every $Y \in |\mathbf{Comp}|$ and compact metric space (X, d_X) the map

$$F\colon C(Y, X) \to C(FY, FX), \enspace f \mapsto Ff,$$

is an isometric embedding.

Recall that a map of metric spaces $f\colon (X, d_X) \to (X', d_{X'})$ is called a *contraction* if there exists a positive $c < 1$ such that $d'(f(x), f(y)) \le cd(x, y)$ for all $x, y \in X$ (this c is said to be a *ratio* of the contraction f.

Now, let (X, d_X) be a separable locally compact metric space and F a metrizable functor. Further, we assume that the functor F preserves contactions, moreover, if f is a contraction with a ratio c, then so is the map $Ff\colon (FX, d_{FX}) \to (X', d_{FX'})$. Represent X as the countable union of an increasing family of its compact subspaces, $X = \cup_{i=1}^\infty X_i$, then $X = \varinjlim X_i$. Denote by $(F_{\mathfrak{M}}X, d_{F_{\mathfrak{M}}X})$ the countable direct limit $\varinjlim_{\mathfrak{M}}(FX_i, d_{FX_i})$ in the category of metric spaces and isometric embeddings. Further, we drop $\mathfrak{M}$ in the notation. From now on, let us suppose that F is the functorial part of a perfectly metrizable monad and F is regular. Suppose that a finite family of contractions $f_0, f_1, \dots, f_{n-1}\colon X \to X$ is given. Fix $a \in Fn$ and for every $b \in FX$ define the map $h_{(b; f_0, f_1, \dots, f_{n-1})}\colon n \to FX$ by the formula $h_{(b; f_0, f_1, \dots, f_{n-1})}(i) = Ff_i(b)$. Finally, define the map $\Phi\colon FX \to FX$ by the formula

$$\Phi(b) = \mu X(Fh_{(b; f_0, f_1, \dots, f_{n-1})})(a), \enspace b \in FX.$$

Proposition 3.9.15. *In the above assumptions, there is a unique $b \in FX$ such that $b = \Phi(b)$.*

Proof. Note that this fact is well-known for $F = \exp$; see, e. g., K. J. Falconer [1985]. Apply this to the set $\operatorname{supp}(b)$. Let $\Psi(A) = \cup_{i \in n} f_i(A)$ for $A \in \exp X$, $\Psi_0 = \Psi$, $\Psi^{i+1} = \Psi \circ \Psi^i$ for $i > 0$. We see that the closure

Y of the set $\bigcup_{i\in\omega}\Psi^i(\mathrm{supp}(b))$ is compact. Thus, without loss of generality, we may suppose that X is compact. We are going to show that Φ is a contraction. Indeed, let $b', b'' \in FX$. Since the maps Ff_i are contractions with ratio c,

$$\begin{aligned} d(h_{(b';f_0,f_1,\dots,f_{n-1})}(i), &h_{(b'';f_0,f_1,\dots,f_{n-1})}(i)) \\ &= d(Ff_i(b'), Ff_i(b'')) < cd(b', b''). \end{aligned}$$

Thus,

$$d^{\sup}(Fh_{(b';f_0,f_1,\dots,f_{n-1})}, Fh_{(b'';f_0,f_1,\dots,f_{n-1})}) < cd(b', b''),$$

and since μX is nonexpanding, we are done.

By the Banach contraction principle, there exists a unique fixed point of the map Φ. □

Exercises

1. The *Gromov-Hausdorff distance* $\varrho_{GH}((X_1, d_1), (X_2, d_2))$ is defined as the infimum of the Hausdorff distances between the images of (X_1, d_1) and (X_2, d_2) under isometric embeddings into a metric space. The obtained metric space of (the isometry classes of) the metric compacta is denoted by $\mathfrak{MC}$. Any metrization of a (weakly, almost) normal functor determines a map $(X, d_X) \mapsto (FX, d_{FX})$ in $\mathfrak{MC}$. Prove that this map is continuous.
2. Is the functor $\mathrm{cc}\,P$ uniformly metrizable? Is the monad $(\mathrm{cc}\,P, \eta, \psi)$ (see Exercise ?? Section 3.1) perfectly metrizable?
3. Let (X, d) be a compact metric space. Define the function $\hat{d}\colon \lambda X \times \lambda X \to \lambda X$ by the formula:

$$\hat{d}(\mathcal{M}, \mathcal{N}) = \inf\{\varepsilon > 0 \mid \text{ for every } M \in \mathcal{M} \text{ there exists } N \in \mathcal{N},\ N \subset O_\varepsilon(M)\}.$$

Prove that $\hat{d} \equiv \hat{d}_X$ is a metric on λX and this metric determines the structure of perfectly metrizable monad for the superextension monad $\mathbb{L}$.
4. Show that the monads $\mathbb{N}_k$ are perfectly metrizable.

3.10. Notes and comments to Chapter 3

It is hard to say who invented the hyperspace monad (see O. Wyler [1981]). Theorem 3.2.3 is due to V. Fedorchuk [1988]. The functional representation of the hyperspace monad (Theorem 3.2.5) is obtained by T. Radul [1997].

The superextension monad and the inclusion hyperspace monad were investigated respectively by M. Zarichnyi [1987b] and T. Radul [1990a].

The probability measure monad was considered by T. Świrszcz [1984].

Corollary 3.3.7 is due to A. Teleiko [1995]. For Proposition 3.4.2 see M. Zarichnyi [1991c]. Theorems 3.4.4 and 3.4.5 are proved by M. Zarichnyi [1986].

Theorem 3.5.2 and Corollary 3.5.5 are obtained by M. Zarichnyi [1990b].

Theorems 3.5.6 and 3.5.8 are proved by M. Zarichnyi ([1992b] and [1992a], respectively). For Theorems 3.5.17 and 3.5.18 see A. Teleiko ([1998] and [1996] respectively). The results of Subsection 3.6.3 are obtained by M. Zarichnyi [1987b].

The results of Subsection 3.6.4 are proved by T. Radul [1990a]. The results of Subsection 3.6.5 are proved by T. Radul [1990b].

Theorems 3.7.5 and 3.8.3 are due to M. Zarichnyi [1991b]. The criterion of barycentric openness (Theorem 3.6.9) is obtained by V. Fedorchuk [1992] (Lemma 3.6.7 is proved by N. Bourbaki).

The notion of perfectly metrizable monad is introduced by V. Fedorchuk [1990]. This notion axiomatized the construction of iterated hyperspace functor by H. Toruńczyk and J. West [1983] and the proof of Proposition 3.9.4 follows that of the corresponding fact in H. Toruńczyk and J. West [1983]. Theorem 3.9.5 is a generalization of the corresponding result from M. Zarichnyi [1986b] for the superextension functor.

The metric on the space of probability measures is defined by V. Uspenskii [1990] (see also J. E. Hutchinson [1981]).

Chapter 4.

Geometric properties of functors

This Chapter is devoted to numerous connections between the general theory of functors and monads and geometric topology. In Section 4.1 we consider the problem of preservation of finite-dimensional spaces and manifolds by functors of finite degree. The main result of Section 4.2 is the Basmanov theorem on preservation of manifolds modeled on the Hilbert cube (Q-manifolds) by functors of finite degree. The class of LC^n-spaces is the n-dimensional counterpart of that of ANR-spaces and in Section 4.3 we deal with the problem of preservation of LC^n-spaces by functors.

Some interactions between functors and equivariant topology are studied in Section 4.4. Section 4.5 is concerned with some homotopy an shape properties of of functors.

4.1. Preservation of finite-dimensional spaces and manifolds

4.1.1. Preservation of finite-dimensional spaces

Theorem 4.1.1. *Let F be a monomorphic functor that preserves finite intersections and empty set, $\deg(F) = n$. If $\dim Fn < \infty$, then F preserves the class of finite-dimensional compact Hausdorff spaces. Moreover, $\dim FX \leq n \dim X + \dim Fn$.*

Proof. Using induction by k, we show that

$$\dim F_k X \le k \dim X + \dim Fk. \tag{4.1}$$

If $k = 1$, then $F_1 X$ is homeomorphic to the product $X \times F1$ and 4.1 holds. Suppose that we have 4.1 with k replaced by $k-1$. The space $F_k X$ is locally homeomorphic to open subset of the product $X^k \times Fk$ at all points of the set $F_k X \setminus F_{k-1} X$. Then 4.1 holds by the sum theorem for dimension and the Dowker theorem. □

Theorem 4.1.2. *If the conditions of Theorem 4.1.1 hold and F preserves preimages, then F preserves the class of maps of compact Hausdorff spaces having finite-dimensional fibers.*

Proof. This follows from Theorem 4.1.1 and the inclusion

$$Ff^{-1}(a) \subset Ff^{-1}(\operatorname{supp} a) = F(f^{-1}(\operatorname{supp} a))$$

for any morphism $f\colon X \to Y$ in **Comp** and $a \in FY$. □

4.1.2. Preservation of finite-dimensional manifolds

The *n-dimensional complex projective space* $\mathbb{CP}^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\mathbb{C} \setminus \{0\}$, the equivalence classes are denoted by

$$(a_0 : a_1 : \cdots : a_n) = \{(ca_0, ca_1, \ldots, ca_n) \mid c \in \mathbb{C} \setminus \{0\}\}.$$

Theorem 4.1.3. *The space $SP^n S^2$ is homeomorphic to $\mathbb{CP}^n$ for all $n \in \mathbb{N}$.*

Proof. Note that S^2 is the one-point compactification $\bar{\mathbb{C}}$ of the complex plane $\mathbb{C}$ (with ∞ as compactifying point). Define $f\colon \bar{\mathbb{C}}^n \to \mathbb{CP}^n$ by

$$f(z_1, z_2, \ldots, z_n) = (a_0 : a_1 : \cdots : a_n),$$

where $z_1, \ldots, z_n$ are the n roots of the equation $\sum_{i=0}^n a_i x^i = 0$ (if such an equation is of degree $n-k$, then we assume that ∞ is its root with multiplicity k). The map f factors through the quotient map $p\colon \bar{\mathbb{C}}^n \to SP^n \bar{\mathbb{C}}$, $f = h \circ p$, where h is, obviously, a homeomorphism.

We left the details to the reader. □

Corollary 4.1.4. *The functor SP^n preserves the class of 2-dimensional manifolds.*

Theorem 4.1.5. *If M is an m-dimensional manifold, $m \geq 2$, and $n \geq 3$, then the space $\exp_n M$ is not a manifold.*

Proof. Without restricting generality we may assume that M is connected. For every $A \in \exp_n M$ there exists an open subset U of M homeomorphic to $\mathbb{R}^m$ such that $A \in \langle U \rangle \subset \exp_n M$. Hence, it is sufficient to prove that $\exp_n \mathbb{R}^m$ is not a manifold.

Choose distinct points $x_1, \ldots, x_{n-1}$ in $\mathbb{R}^m$. For $\varepsilon > 0$ let

$$V_\varepsilon = \{A \in \exp_n \mathbb{R}^m \mid d_H(A, \{x_1, \ldots, x_{n-1}\} < \varepsilon\}$$

(d_H is the Hausdorff metric induced by the euclidean metric d on $\mathbb{R}^m$). We suppose that ε is so small that the ε-neighborhoods $O_\varepsilon(x_i)$ of the points x_i are disjoint. Then V_ε is a connected open subset in $\exp_n \mathbb{R}^m \setminus \exp_{n-1} \mathbb{R}^m$. Let $W = V_\varepsilon \cap \setminus \exp_{n-1}$, then $V_\varepsilon \setminus W$ is the disjoint union of open subsets $C_i = \{A \in V_\varepsilon \setminus W \mid |a \cap O_\varepsilon(x_i)| = 2, i = 1, \ldots, n-1$.

By our assumption, V_ε is a connected open subset in mn-dimensional manifold. But since $\dim W = m(n-1) \leq mn - 2$, we see that V_ε is separated by a subset of codimension ≥ 2, which gives a contadiction (see R. Engelking [1978]). □

Exercises

1. Show that $\exp_3 S^1$ is homeomorphic to S^3.
2. Show that $\lambda_3 S^1$ is homeomorphic to S^3.
3. Show that $\exp_2([0,1]^2)$ is homeomorphic to $[0,1]^4$.

4.2. Preservation of ANR-spaces and manifolds

A metrizable space X is called *an absolute retract* (AR-space) if for every closed embedding f of X into a metrizable space Y the subspace $f(X)$ is a retract of Y. A metrizable space X is called *an absolute neighborhood retract* (briefly ANR-space) if for every closed embedding f of X into a metrizable space Y the subspace $f(X)$ is a retract of some its neighborhood in Y.

The class of AR-spaces (ANR-spaces) is denoted by AR (respectively ANR). See J. van Mill [1989] for the properties of ANR-spaces.

In this section we consider the problem of preserving compact AR-spaces (ANR-spaces by some functors of finite degree. For this purpose we need some propertiers of compact A(N)R-spaces related to the theory of infinite-dimensional manifolds.

Recall that two maps $f, g \colon X \to Y$ are called *$\mathcal{U}$-homotopic*, where $\mathcal{U}$ is a cover of Y, if there exists a homotopy $H \colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$, $x \in X$, and for every $x \in X$ there is $U \in \mathcal{U}$ such that $H(\{x\} \times [0,1]) \subset U$ (such a homotopy is called a *$\mathcal{U}$-homotopy*). A map $f \colon X \to Y$ is called a *fine homotopy equvalence* provided that there exists a map $g \colon Y \to X$ such that for every open cover $\mathcal{U}$ of Y the maps $f \circ g$ and id_Y are $\mathcal{U}$-homotopic and the maps $g \circ f$ and 1_X are $f^{-1}(\mathcal{U})$-homotopic. The following is easy to prove.

Lemma 4.2.1. *A retraction $g \colon A \to B$ is a fine homotopy equivalence if and only if for every open cover $\mathcal{U}$ of A the map $g \colon A \to A$ is $f^{-1}(\mathcal{U})$-homotopic to 1_A.*

We will need the following result of D. W. Curtis [1981].

Theorem 4.2.2. *Let $X = \varprojlim\{X_i, p_i\}$ be the inverse limit of an inverse sequence, where every X_i is a compact metrizable* ANR*-space and every p_i is a fine homotopy equivalence. Then $X \in$* ANR.

Theorem 4.2.3. *Let F be a continuous, monomorphic functor that preserves finite intersections and empty set, $\deg(F) = n$. If Fn is a finite-dimensional compact metrizable space and $Fk \in$* ANR *for every $k = 1, \dots, n$, then F preserves the class of (finite-dimensional) compact metrizable* ANR*-spaces.*

Proof. Suppose that X is a finite-dimensional compact metrizable ANR-space. By induction, prove that $F_k X \in$ ANR. This is the case for $k = 1$, because $F_1 X$ is homeomorphic to $X \times F1$. Suppose that we have already proved that $F_{k-1} X \in$ ANR. Define the action ψ of the symmetric group S_k on $Z = C(k, X) \times Fk$ by the following manner:

$$\psi(\sigma)(f, a) = (f \circ \sigma^{-1}, F\sigma(a)),\ f \in C(k, X),\ a \in Fk,\ \sigma \in S_k.$$

Recall that the map $\pi_{F,X,k} \colon C(k, X) \times Fk \to FX$, $\pi_{F,X,k}(f, a) = Ff(a)$, where $f \in C(k, X)$, $a \in Fk$, is continuous (see Proposition 2.6.6). Let $Z' = \pi^{-1}_{F,X,k}(F_{k-1} X)$ and $\pi' = \pi_{F,X,k}|Z'$. Denote by T the

orbit space of Z with respect to the action of S_k and by $\gamma\colon Z \to T$ the quotient map. Since $\pi_{F,X,k} \circ \psi(\sigma) = \pi_{F,X,k}$ for every $\sigma \in S_k$, there exists a map $\varrho\colon T \to F_k X$ such that $\varrho \circ \gamma = \pi_{F,X,k}$. Let $T' = \varrho^{-1}(F_{k-1}X)$ and $\varrho' = \varrho|T'$. Then the space $F_k X$ is homeomorphic to the space obtained from the spaces T and $F_{k-1}X$ by glueing along the map $\varrho'\colon T' \to F_{k-1}X$. It is sufficient to prove that $T, T', F_{k-1}X \in \mathrm{ANR}$ (see J. van Mill [1989]). Since the orbit space of finite-dimensional compact metrizable ANR-space with respect to action of a finite group is an ANR-space, we have to prove that $Z, Z' \in \mathrm{ANR}$. Since Z is a product of compact metrizable ANR-spaces, we see that $Z \in \mathrm{ANR}$ (see J. van Mill [1989]). Since the functor F is monomorphic, we have

$$Z' = H \cup \bigcup\{H_{\alpha\beta} \mid \alpha, \beta \in k,\ \alpha \neq \beta\},$$

where

$$H = C(k, X) \times F_{k-1}k,\ H_{\alpha\beta} = \{f \mid f \in C(k, X),\ f(\alpha) = f(\beta)\} \times Fk.$$

In order to prove that $Z' \in \mathrm{ANR}$ it is sufficient to verify that the intersection of any subfamily of the family $\{H\} \cup \{H_{\alpha\beta} \mid \alpha, \beta \in k,\ \alpha \neq \beta\}$ is an ANR-space (see J. van Mill [1989]). But every such an intersection is homeomorphic to the product of some power of the space X onto one of the spaces Fk, $F_{k-1}k$ and, therefore, is an ANR-space.

We have already proved that F preserves the class of finite-dimensional ANRs. Consider the general case. Embed the space $X \in \mathrm{ANR}$ into the Hilbert cube $Q = I^\omega$. Then X is a retract of its closed neighborhood of the form $K \times I^\omega$, for some finite polyhedron K, and it is sufficient to prove that $F(K \times I^\omega) \in \mathrm{ANR}$.

Represent $K \times I^\omega$ as $\varprojlim\{K \times I^n, p_n\}$, where $p_n\colon K \times I^{n+1} \to K \times I^n$ denotes the projection onto the initial factors. Then $F(K \times I^\omega) = \varprojlim\{F(K \times I^n), Fp_n\}$. As we have already proved, $F(K \times I^n) \in \mathrm{ANR}$. In order to be able to apply Theorem 4.2.2, we need to prove that the maps Fp_n are fine homotopy equivalences.

Use Lemma 4.2.1. For every $t \in I$ define the map $h_t\colon K \times I^{n+1} \to K \times I^{n+1}$ by the formula

$$h_t(k, x_0, \ldots, x_n) = (k, x_0, \ldots, tx_n).$$

Obviously, for every open cover $\mathcal{U}$ of $K \times I^n$ the family Fh_t is an $Fp_n^{-1}(\mathcal{U})$-homotopy of the identity mepa and the map Fp_n (the latter is considered as a retraction of $K \times I^{n+1}$ onto $K \times I^n \times \{0\}$).

$\square$

Recall that a space is called *contractible* if its identity map is homotopic to a constant map. It is well-known that AR-spaces are the contractible ANR-spaces.

Lemma 4.2.4. *Let F be a functor in* **Comp** *satisfying the conditions of Theorem 4.2.3 and the space $F1$ is contractible. Then F preserves the class of contractible spaces.*

Proof. Let $H: X \times [0,1] \to X$ be a homotopy of 1_X and a constant map. For every $t \in [0,1]$ denote by $h_t: X \to X$ the map defined by the formula $h_t(x) = (x,t)$, $x \in X$. Then the map $\tilde{H}: FX \times [0,1] \to FX$, $\tilde{H}(a,t) = Fh_t(a)$ is a homotopy of 1_{FX} and a retraction onto $\{x_0\}$, for some $x_0 \in X$. Since $F1$ is contractible, the this reraction is homotopic to a constant map. $\square$

Theorem 4.2.5. *Suppose a functor F satisfies the conditions of Theorem 4.2.3. Then F preserves the class of (finite-dimensional) compact metrizable* AR*-spaces.*

Proof. This follows from Theorem 4.2.3 and the fact that the functor F preserves the class of contractible spaces. $\square$

Recall that $Q = I^\omega$ is the Hilbert cube. A compact metrizable space is called a *Q-manifold* if it has a base consisting of subsets homeomorphic to open subsets of Q.

Theorem 4.2.6 (ANR-theorem). *For every compact* ANR*-space X the product $X \times Q$ is a Q-manifold.*

Corollary 4.2.7. *A compact space X is an absolute neighborhood retract iff X is a retract of a Q-manifold.*

The corresponding results are valid for AR-spaces.

Theorem 4.2.8. *A compact metrizable space X is an* AR*-space if and only if $X \times Q$ is homeomorphic to Q.*

Theorem 4.2.9. *(***Toruńczyk Characterization Theorem***) A compact metrizable* ANR*-space (respectively* AR*-space) is a Q-manifold (is homeomorphic to Q) if and only if the identity map 1_X can be uniformly approximated by maps with disjoint images.*

Theorem 4.2.10. *Suppose that a monomorphic continuous functor F in the category* **Comp** *preserves weight, intersections and empty sets. If F preserves metrizable compact ANR-spaces (respectively AR-spaces), then F preserves the class of compact Q-manifolds (respectively FQ is homeomorphic to Q).*

Proof. The proof follows from Toruńczyk Characterization Theorem. Indeed, if X is a Q-manifold, then there exist two sequences of maps $(f_i\colon X \to X)$, $(g_i\colon X \to X)$ such that $\lim_{i\to\infty} f_i = \lim_{i\to\infty} g_i = 1_X$ and $f_i(X) \cap g_i(X) = \varnothing$. Applying F to these maps, we see that $1_{FX} = \lim_{i\to\infty} Ff_i = \lim_{i\to\infty} Fg_i$ and $Ff_i(FX) \cap Fg_i(FX) \subset F(f_i(X) \cap g_i(X)) = F\varnothing = \varnothing$. □

4.2.1. Functors and 1-soft maps

Let $\pi\colon Q \times Q \to Q$ denote the projection onto the first factor.

Theorem 4.2.11. *Let F be a finite normal functor such that the map $F\pi\colon F(Q \times Q) \to FQ$ is 1-soft. Then F is a power functor.*

Proof. The map $F\,\mathrm{pr}_1$, being 1-soft, is open and by Proposition 2.10.20, the functor F is bicommutative.

Show that F is a power functor; for this it is sufficient to prove that for every $a \in Fn$, $\deg(a) = n$, $n < \omega$, and for every maps $f, g\colon n \to n$ we have $Ff(a) \neq Fg(a)$, whenever $f \neq g$.

Suppose the contrary and let $j\colon n \to Q$ be an embedding, $i_k\colon n \to Q$, $k \in \mathbb{N}$, a sequence of embeddings converging to the constant map $i\colon n \to \{x_0\}$, for $x_0 \in Q$.

Let

$$b_k = F(i_k, j \circ f)(a), c_k = F(i_k, j \circ g)(a), \;\; k \in \mathbb{N}.$$

Then, obviously, $F\,\mathrm{pr}_1(b_k) = F\,\mathrm{pr}_1(c_k)$ and

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = F(i, j \circ f)(a) = F(i, j \circ g)(a).$$

Let

$$\begin{aligned} Z =&\{(s,t) \in \mathbb{R}^2 \mid s = 1/k,\; 0 \le t_k \le 1/k,\; k \in \mathbb{N}\} \cup \{(0,0)\} \\ A =&\{(s,t) \in Z \mid t \in \{0, 1/k\},\; k \in \mathbb{N}\}. \end{aligned}$$

Define the maps $\varphi\colon A \to F(Q \times Q)$, $\psi\colon Z \to FQ$ by the formulae

$$\begin{aligned} \varphi(1/k, 0) &= b_k, \ \varphi(1/k, 1/k) = c_k, \ k \in \mathbb{N}, \\ \varphi(0,0) &= F(i, j \circ f)(a) = F(i, j \circ g)(a), \\ \psi(1/k, t) &= Fi_k(a), \ k \in \mathbb{N}, \\ \psi(0,0) &= \eta Q(x_0). \end{aligned}$$

(continuity of ψ easily follows from the fact that $\operatorname{supp}(\eta Q(x_0)) = \{x_0\}$).

By 1-softness of $F\operatorname{pr}_1$, there exists a map $\Phi\colon Z \to F(Q \times Q)$ such that $F\operatorname{pr}_1 \circ \Phi = \psi$ and $\Phi|A = \varphi$.

Let $\operatorname{supp}(F(i, j \circ g)(a)) = \{y_1, \dots, y_m\}$ and $U_1, \dots, U_m$ be disjoint neighborhoods of the points $y_1, \dots, y_m$. Since finite functors are those with continuous supports (Corollary 2.4.6), without loss of generality, we may assume that $\operatorname{supp}\Phi(z) = U_1 \cup \dots \cup U_m$ for every $z \in Z$.

We assume that $U_i = W \times V_i$ for some neighborhood W of the point x_0 and V_i is a neighborhood of $y_i = (x_0, z_i)$, $1 \le i \le n$. Let $r\colon U_1 \cup \dots \cup U_m \to Q$ be a map acting by the formula $r(x, z) = (x, z_i)$, whenever $(x, z) \in U_i$, $1 \le i \le m$. Then

$$\lim_{k\to\infty} Fr(b_k) = \lim_{k\to\infty} Fr(c_k) = F(i, j \circ f)(a).$$

Since $f \neq g$, we see that

$$\operatorname{supp} Fr \circ F(i_k, j \circ f) \neq \operatorname{supp} Fr \circ F(i_k, j \circ g)$$

and

$$\operatorname{supp} Fr \circ \Phi(1/k, t) \subset i_k(n) \times \{y_1, \dots, y_m\}$$

for every $t \in [0, 1/k]$. It follows from finiteness of F that the element $Fr \circ \Phi(1/k, t)$ fails to depend on t and we obtain a contradiction. □

Theorem 4.2.12. *Let F be a normal functor of finite degree n. Then $F \cong (-)^n$ if and only if the map $F\pi\colon F(Q \times Q) \to FQ$ is 1-soft.*

Proof. We have only to show the sufficiency. Define the action of the n-symmetric group S_n on the space $Z = Fn \setminus F_{n-1}n$ by the following manner: $\sigma_{F,n}(a) = F\sigma(a)$ for $\sigma \in S^n$ and $a \in Z$.

Denote by $\pi_i\colon n_1 \times n_2 \to n_i$, where $i = 1, 2$ and $n_i = n$, the projection on the i-th factor. Show that the diagonal product $(F\pi_1, F\pi_2)$ is surjective. Indeed, since $F\pi$ is 1-soft, it is open. Hence, the functor F

is bicommutative (see Proposition 2.10.20). Thus, being a characteristic map of the following bicommutative diagram

$$\begin{tikzcd} F(n_1 \times n_2) \arrow[r, "F\pi_2"] \arrow[d, "F\pi_1"'] & Fn_2 j_2 \arrow[d, "F"] \\ Fn_1 j_1 \arrow[r, "F"] & F1 = 1 \end{tikzcd}$$

(here $j_i\colon n_i \to 1$ is a constant map), the map $(F\pi_1, F\pi_2)\colon F(n_1 \times n_2) \to Fn_1 \times Fn_2$ is surjective.

Let $x \in Fn_1$ and $y \in Fn_2$ be points such that $\mathrm{supp}(x) = n_1$, $\mathrm{supp}(y) = n_2$. Consider the map $F\pi_1$. By Proposition 2.6.6 we have

$$\pi_{F,n}(n_1 \times n_2)^{-1}((F\pi_1)^{-1}(x)) = \bigcup\{M_\varphi \mid \varphi \in n_1^n\},$$

where

$$M_\varphi = \{\xi \mid \xi \in (n_1 \times n_2)^n,\ \pi_1 \circ \xi(i) = \varphi(i) \text{ at } i \in \varphi^{-1}(n_1)\} \times (F\varphi)^{-1}(x).$$

and $\pi_{F,n}(n_1 \times n_2)\colon (n_1 \times n_2)^n \times Fn \to F(n_1 \times n_2)$ is the Basmanov map. Since $\deg(x) = n$, all maps $\varphi\colon n \to n_1$ are homeomorphisms. Therefore, the maps $F\varphi$ are also homeomorphisms. For every $x \in Z$ define the set

$$\mathrm{Orb}_{\pi_1}(x) = \{(F\varphi)^{-1}(x) \mid \varphi\colon n \to n_1 \text{ is a homeomorphism}\}.$$

It is easy to verify that this set is an orbit of some point of Z with respect to the action of S_n. Show that

$$\mathrm{Orb}_{\pi_1}(x) = \{c \in Z \mid \text{there exits an embedding } \xi\colon n \to n_1 \times n_2 \\ \text{with } F\pi_1 \circ F\xi(c) = x\}.$$

Indeed, if $c \in \mathrm{Orb}_{\pi_1}(x)$, then there exists a homeomorphism $\psi\colon n \to n_1$ such that $c = (F\psi)^{-1}(x)$. Define a map $\xi\colon n_1 \times n_2$ by the condition: $\xi(i) \in \pi_1^{-1}(\psi(i))$ for all $i \in n$. Then the map ξ is an embedding and

$$F\pi_1 \circ F\xi(c) = F(\pi_1 \circ \xi)(c) = F\psi(c) = x.$$

Now suppose that there exists an embedding $\xi\colon n \to n_1 \times n_2$ with $F\pi_1 \circ F\xi(c) = x$, but $c \notin \mathrm{Orb}_{\pi_1}(x)$. Then since $\deg(x) = n$, we see that the map $\psi = \pi_1 \circ \xi$ is surjective, and hence, it is a homeomorphism. Therefore, $c = (F\psi)^{-1}(x)$, a contradiction.

Similarly, we define a set $\mathrm{Orb}_{\pi_2}(y)$, which is an orbit of some point of Z and satisfies the equality

$$\mathrm{Orb}_{\pi_2}(y) = \{c \in Z \mid \text{there exits an embedding } \xi\colon n \to n_1 \times n_2 \\ \text{with } F\pi_2 \circ F\xi(c) = y\}.$$

Since $(F\pi_1, F\pi_2)$ is a surjective map, there exists an element $r \in (F\pi_1, F\pi_2)^{-1}(z)$, where $z = (x, y)$. By $\deg F = n$, we have $|\operatorname{supp}(r)| \leq n$. Moreover, $|\operatorname{supp}(r)| = n$, because $F\pi_1(r) = x$ and $F\pi_2(r) = y$. It is easy to show that there exist a point $c \in Z$ and an embedding $\xi\colon n \to n_1 \times n_2$ such that $\pi_{F,n}(n_1 \times n_2)(\xi, c) = r$. Hence, $c \in \mathrm{Orb}_{\pi_1}(x) \cap \mathrm{Orb}_{\pi_2}(y)$ and thus, $\mathrm{Orb}_{\pi_1}(x) = \mathrm{Orb}_{\pi_2}(x)$. Since, moreover, $\deg(x) = \deg(y) = n$, every two orbits of Z have nonempty intersection. Consequently, Z has a unique orbit, and therefore, $|Z| \leq n!$.

Using finiteness of $Fn \setminus F_{n-1}n$ and the equality $\deg F = n$, it is easy to show that the space $F(n_1 \times n_2) \setminus F_{n-1}(n_1 \times n_2)$ is fnite. Fix a point $x_0 \in Fn_1$ with $\operatorname{supp}(x_0) = n_1$. For every $y \in Fn_2$ there exists a point $q \in F(n_1 \times n_2)$ such that $(F\pi_1, F\pi_2)(q) = (x_0, y)$. Since $F\pi_1(q) = x_0$ and $\deg(x_0) = n$, we have that $\deg(q) = n$. Moreover, $F\pi_2(q) = y$. Hence, $F\pi_2|\big(F(n_1 \times n_2) \setminus F_{n-1}(n_1 \times n_2)\big)$ is a surjective map. Therefore, the space Fn is finite.

Thus, F is finite and we may apply Theorem 4.2.11. Thus, F is a power functor and, since $\deg F = n$, we see that $F \cong (-)^n$. □

4.2.2. Functors of finite degree and extension property

The *Kuratowski notation* $Y\tau X$ means that that every map $g\colon A \to X$ defined on a closed subset A of a space Y can be extended to a map $\bar{g}\colon Y \to X$.

For a map $f\colon X \to X'$ the notation $Y\tau f$ means that for any map $g\colon A \to X$ defined on a closed subset A of a space Y there is a map $\bar{g}\colon Y \to X'$ such that $fg = \bar{g}|A$. Evidently, $Y\tau X$ is equivalent to $Y\tau 1_X$.

By an ANR-sequence we understand an inverse system $\mathcal{S} = \{X_i, p_i^j\}$ over positive integers such that $X_i \in \mathrm{ANR}$. Suppose that $\mathcal{S} = \{X_i, p_i^j\}$ is an ANR-sequence. For a space Y by $Y\tau\mathcal{S}$ we denote the following property: for every i there is $j \geq i$ such that $Y\tau p_i^j$.

Obviously, if $l \leq i$ and $k \geq j$, then $Y\tau p_i^j$ implies $Y\tau p_l^k$.

Theorem 4.2.13. *If $Y\tau\mathcal{S}$, then $Y\tau SP^n(\mathcal{S})$.*

Proof. Fix i and define a sequence $i = i_0 \leq i_1 \leq \cdots \leq i_n$ such that for every $\alpha = 0, 1, \ldots, n-1$ we have $Y \tau p^{i_\alpha}_{i_{\alpha+1}}$. Show that $Y \tau SP^n(p^j_i)$, where $j = i_n$.

Let $A \subset Y$ be a closed subset and $f \colon A \to SP^n(X_j)$ a map. We consider X_j as a closed subset of an AR-space L and extend the map f to a map $g \colon Y \to SP^n(L)$ (the functor SP^n preserves the class of AR-spaces (see Theorem 4.2.5). The set

$$\Gamma_g = \{(y, l) \in Y \times L \mid l \in \operatorname{supp}(g(y))\}$$

is the graph of the SP^n-valued map g. Let $\pi_1 \colon \Gamma_g \to Y$ and $\pi_2 \colon \Gamma_g \to L$ be the restrictions onto Γ_g of the projections. Let $M = \pi_1^{-1}(A)$ and $Y_\alpha = \pi_1^{-1}(\{y \in Y \mid |\operatorname{supp}(g(y)) \leq \alpha\})$, $\alpha = 1, \ldots, n$. There exists an extension $h_n \colon U \to X_{i_n}$ of the map $\pi_2|M \colon M \to X_{i_n}$ onto a closed neighborhood U of M in Γ_g. Since Y_1 is homeomorphic to a closed subset of Y, we have $Y_1 \tau p^{i_n}_{i_{n-1}}$ and there exists a map $\tilde{h}_{n-1} \colon Y_1 \to X_{i_{n-1}}$ such that $\tilde{h}_{n-1}|(U \cap Y_1) = p^{i_n}_{i_{n-1}}|(U \cap Y_1)$. The maps $\tilde{h}_{n-1}$ and $p^{i_n}_{i_{n-1}} h_n$ coincide on the intersection of their domains and, thus, determine the map $\bar{h}_{n-1} \colon U \cup Y_1 \to X_{n-1}$. There exists an extension $h_{n-1} \colon U_1 \to X_{n-1}$ of the map $\bar{h}_{n-1}$ onto a closed neighborhood U_1 of the set $M \cup Y_1$ in Γ_g.

Note that $Y_2 \setminus \operatorname{Int} U_1$ is the discrete sum of subsets homeomorphic to closed subsets of Y. Thus, $Y_2 \setminus \operatorname{Int} U_1 \tau p^{n-1}_{n-2}$ and, consequently, there exists a map $\tilde{h}_{n-2} \colon Y_2 \setminus \operatorname{Int} U_1 \to X_{i_{n-2}}$ such that $\tilde{h}_{n-2}|(U_1 \cap Y_2) = p^{i_{n-1}}_{i_{n-2}}|(U_1 \cap Y_2)$. As above, the maps $\tilde{h}_{n-2}$ and $p^{i_{n-1}}_{i_{n-2}} h_n$ coincide on the intersection of their domains and, thus, determine the map $\bar{h}_{n-2} \colon U \cup Y_2 \to X_{i_{n-2}}$.

Proceeding in this way, we will find a map $h_0 \colon U \cup Y_n = \Gamma_g \to X_{i_0}$ such that $h_0|M = p^{i_n}_{i_0} \pi_2$.

Finally, define the map $G \colon Y \to SP^n X_{i_0}$ by the formula: $G(y) = SP^n(h_0)(g(y))$, $y \in Y$. It is easy to see that $G|A = SP^n(p^j_i)f$. □

Corollary 4.2.14. *If $Y \tau X$ and X is a compact metrizable* ANR*-space, then $Y \tau SP^n X$.*

Exercise

1. Generalize Theorem 4.2.13 and Corollary 4.2.14 onto the class of normal functors of finite degree (with finite supports) that preserve compact metrizable ANR-spaces.

4.3. Preservation of LC^n-spaces

The notion of ANR-space has its counterpart in the class of $\leq n$-dimensional spaces.

A metric space X is called an *absolute (neighborhood) extensor in dimension* n if for every map $f\colon A \to X$ defined on a closed subset A of a metric space Y with $\dim Y \leq n$, there exists an extension of f onto the whole space Y (respectively, onto a neighborhood of A in Y).

The class of absolute (neighborhood) extensors in dimension n is denoted by A(N)E(n). By the Kuratowski-Dugundji theorem (see, e. g., K. Borsuk [1967]), the class AE(n) (respectively, ANE(n)) coincides with the class of LC^{n-1}- and C^{n-1}-spaces (respectively, LC^{n-1}-spaces). Recall that a space X is called an LC^{n-1}*-space* if for every point $x \in X$ and every neighborhood U of x there exists a neighborhood V of X satisfying the condition: For every map $f\colon S^k \to V$, where $k \leq n-1$, there exists an extension $\bar{f}\colon B^{k+1} \to U$ of f. A space X is called a C^{n-1}*-space* if for every map $f\colon S^k \to X$, where $k \leq n-1$, there exists an extension $\bar{f}\colon B^{k+1} \to X$ of f.

A map $f\colon X \to Y$ is called n*-invertible* if for every map $g\colon Z \to Y$, where $\dim Z \leq n$, there exists a map $\bar{g}\colon Z \to X$ such that $f \circ \bar{g} = g$.

Theorem 4.3.1. (A. Dranishnikov) *A compact metrizable space is an* AE(n)*-space (respectively* ANE(n)*-space) if it is an* n*-invertible image of* Q *(respectively* Q*-manifold).*

Theorem 4.3.2. *Let* F *be a normal functor with finite supports. If* F *preserves the class of compact metrizable* ANR*-spaces, then* F *preserves the class of compact metrizable* A(N)E(n)*-spaces.*

We need the following auxilary result.

Lemma 4.3.3. *Let* F *be a normal functor with finite supports. Then* F *preserves the class of* n*-invertible maps.*

Proof. Suppose a map $f\colon X \to Y$ is n-invertible. Given a map $g\colon Z \to Y$, where $\dim Z \leq n$, put

$$\Gamma_g\{(x,y) \mid y \in \operatorname{supp}_F(g(x))\} \subset Z \times Y.$$

Note that, since the map supp_F is lower-semicontinuous, the restriction $\operatorname{pr}_1 |\Gamma_g$ of the projection map $\operatorname{pr}_1\colon Z \times Y \to Z$ is open, and, since the map $\operatorname{pr}_1 |\Gamma_f$ is finite-to-one, we have $\dim \Gamma_g \leq n$.

By n-invertibility of f, there exists a map $h\colon \Gamma \to X$ such that $f \circ h = \mathrm{pr}_2|\Gamma_g$ (by $\mathrm{pr}_2\colon Z \times Y \to Y$ we denote the projection map). Define the map $H\colon Z \to FX$ by the following manner. Let $z \in Z$. There exists $a \in Fn$ and a map $\alpha\colon n \to Y$ such that $g(z) = F\alpha(a)$.

Denote by $\alpha'\colon n \to Z \times Y$ the map defined by the formula $\alpha'(i) = (z, \alpha(i))$. Finally, put

$$H(z) = F(h \circ \alpha')(a).$$

(note that we formally used the extension of F onto the category **Tych** in this formula, because the subspace Γ_g needs not be compact).

Then

$$Ff(H(z)) = F(f \circ h \circ \alpha')(a) = F(\mathrm{pr}_2 \circ \alpha')(a) = \alpha(a) = g(z).$$

□

Now the proof of theorem follows from the lemma and the Dranishnikov theorem cited above.

Remark 4.3.4. The given proof of Theorem 4.3.2 does not work outside the class of normal functors. In the denotation of the proof, the map $F(H \circ \alpha')$ can be discontinuous if F is not normal.

Let X be a compact metrizable space. Denote by $LC^n(X)$ the set of all nonempty LC^n-subsets of X. The set $LC^n(X)$ is endowed by the topology of *homotopy n-regular convergence* (see Kuratowski [1957]). A sequence $(A_i)_{i\in\omega}$ converges to $A_\omega \in LC^n(X)$ in this topology, whenever it converges to $A_\omega \in LC^n(X)$ in the Vietoris topology and is equi$-LC^n$, i. e. for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every map $f\colon S^k \to A_i$, $k \le n$, with $\mathrm{diam}(f(S^k)) < \eta$ there exists an extension $\bar f\colon B^{k+1} \to A_i$ with $\mathrm{diam}(f(B^{k+1})) < \varepsilon$.

Recall that, for $B \in |\mathbf{Comp}|$, by $\mathbf{Comp}/B$ we denote the category of “compact Hausdorff spaces over B”. The objects of $\mathbf{Comp}/B$ are the maps $f\colon X \to B$ and the set of morphisms from $f\colon X \to B$ to $f'\colon X' \to B$ consists of maps $\alpha\colon X \to X'$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commutative. Given a normal functor F, define the functor

$$F_0\colon \mathbf{Comp}/B \to \mathbf{Comp}/B$$

by the conditions:

$$F_0 f = (\eta B)^{-1} \circ (Ff|((Ff)^{-1}(\eta B(B))), \ (f\colon X \to B) \in |\mathbf{Comp}/B|,$$
$$F_0\alpha = F\alpha|F_0 f, \ \alpha\colon (f\colon X \to B) \to (f'\colon X' \to B).$$

If a normal functor F with finite supports preserves the class of compact metrizable ANR-spaces, then, by Theorem 4.3.2, F preserves the class of compact metrizable LC^n-spaces, i. e. F determines the map $A \mapsto FA\colon LC^n(X) \to LC^n(FX)$.

The following statement is a generalization of Lemma 4.3.3.

Lemma 4.3.5. *Suppose that $\alpha\colon X \to Y$ is a morphism of the category* $\mathbf{Comp}/B$ *which is an n-invertible map. For every normal functor F with finite supports that preserves the class of compact metrizable ANR-spaces the map $F_0\alpha\colon F_0X \to F_0Y$ is n-invertible.*

Theorem 4.3.6. *Let F be a normal functor with finite supports that preserves the class of compact metrizable ANR-spaces. Then the map*

$$A \mapsto FA\colon LC^n(X) \to LC^n(FX)$$

is continuous in the topology of homotopy n-regular convergence.

Proof. Let $(A_i)_{i\in\omega}$ be a sequence in $LC^n(Q)$ converging to $A_\omega \in LC^n(Q)$ and

$$X = \{(a,i) \in Q \times (\omega+1) \mid a \in A_i\}, \ f = \mathrm{pr}_2|X\colon X \to \omega+1.$$

The result of Dranishnikov [1984] shows that then the map f is locally $(n+1)$-soft. It is also proved in Dranishnikov [1984] that there exists an $(n+1)$-invertible map $g\colon Z \to X$, where Z is a compact metrizable space with $\dim Z \le n+1$. There exists an embedding $j\colon Z \to \mu_{n+1} \times (\omega+1)$ such that $\mathrm{pr}_2 \circ j = f \circ g$ (here μ_{n+1} denotes the $(n+1)$-dimensional universal Menger cube; see R. Engelking [1978]). Local $(n+1)$-softness of the map f implies that there exists a compact neighborhood $U \supset j(Z)$ in $\mu_{n+1} \times (\omega+1)$ and a map $r\colon U \to X$ such

that $r \circ j = g$ and $f \circ r = \mathrm{pr}_2 |U$. Without loss of generality we may suppose that there exists a compact μ_{n+1}-manifold $M \subset \mu_{n+1}$ such that

$$j(Z) \subset (\mathrm{Int}\, M) \times (\omega + 1) \subset M \times (\omega + 1) \subset U.$$

Note that the map $r' = r|(M \times (\omega + 1))$ is $(n + 1)$-invertible.

Apply to the maps f, r', and $\mathrm{pr}_2\colon M \times (\omega + 1) \to \omega + 1$ the functor F_0 (in the category **Comp**$/(\omega + 1)$). By Lemma 4.3.5, the map $F_0 r'$ is $(n + 1)$-invertible, and this easily implies that the map $F_0 f$ is locally $(n+1)$-soft. Hence, the sequence $(A_i)_{i \in \omega}$ converges to FA_ω in the space $LC^n(FQ)$. □

Corollary 4.3.7. *Let F be as in Theorem 4.3.6. A map $f\colon X \to Y$ of compact metrizable spaces is n-soft if and only if so is the map $F_0 f\colon F_0 X \to Y$.*

Problem

1. Is there a counterpart of Theorem 4.3.2 for functors of finite degree (with finite supports) that are not necessarily normal? In particular, for the projective power functors? for the functors of "words of length $\leq n$ in the free topological (Abelian) groups?

4.4. Functors and G-ANR-spaces

4.4.1. G-ANR functor spaces

Consider the case of equivariant absolute (neighborhood) retracts.

Let X be a compact Hausdorff G-space for a compact group G, $\alpha\colon G \times X \to X$ a structure map. By $\alpha^{\#}\colon G \to \mathrm{Homeo}(X)$ the adjoint homomorphism is denoted. Here the homeomorphisms group $\mathrm{Homeo}(X)$ is endowed by compact-open topology. For a semi-normal functor F in **Comp** the composition $F \circ \alpha^{\#}\colon G \to \mathrm{Homeo}(FX)$ is the structure map for an action $\overline{\alpha}\colon G \times FX \to GFX$. It is easily to see that thus the endofunctor acting in the category G-**Comp** of compact G-spaces and equivariant maps is defined. This functor is also denoted by F.

A map $\Delta\colon C(Y, X) \to G\text{-}C(Y, X)$ is called *averaging operator* if for every $f \in C(Y, X)$ the maps Δf and f coincide at the points y such that $f(g \cdot y) = g \cdot f(y)$ for all $g \in G$.

Let F be a weakly normal functor, X, Y be G-spaces. Suppose that F generates a monad (F, η, μ). Moreover, suppose that there exists a G-right-invariant point $\mathcal{M} \in FG$, i.e., for every $h \in G$ we have $Fr_h(\mathcal{M}) = \mathcal{M}$, where $r_h \colon G \to G$ is the right shift by h, $r_h(g) = gh$.

Let $f \colon Y \to FX$ be a map. For $y \in Y$ consider the map $p_y \colon G \to FX$, $p_y(g) = g^{-1} \cdot f(g \cdot y)$, $g \in G$. Set

$$\Delta f(y) = \mu \circ Fp_y(\mathcal{M}). \tag{4.2}$$

Proposition 4.4.1. *Formula* (4.2) *determines an averaging operator.*

Proof. Consider a map $f \colon Y \to FX$. To show that Δf is continuous, we have to remark only that for every $Z_1, Z_2 \in \mathcal{M}\mathbf{Comp}$ the map $j_{Z_1,FZ_2} \colon Z_1 \times FZ_2 \to F(Z_1 \times Z_2)$, $j_{Z_1,FZ_2}(z_1, a) = Fi_{z_1}(a)$ (where $i_{z_1}(z_2) = (z_1, z_2)$, $z_2 \in Z_2$), $a \in FZ_2$, $z_1 \in Z_1$, is continuous (see V. Fedorchuk, V. Filippov [1988]).

Show that Δf is equivariant. Let $h \in G$, $y \in Y$. Then $\Delta f(h \cdot y) = \mu \circ Fp_{hy}(\mathcal{M}) = \mu \circ F^2 h \circ Fp_y \circ Fr_h(\mathcal{M})$, where $r_h \colon G \to G$, $r_h(g) = gh$, is the right shift by h. Hence, $\Delta f(h \cdot y) = Fh \circ \mu \circ Fp_y(\mathcal{M}) = h \cdot \Delta f(y)$.

Now suppose that $f(g \cdot y) = g \cdot f(y)$ for $y \in Y$ and all $g \in G$. Then $p_y(G) = \{f(y)\}$ and $Fp_y(FG) = \{f(y)\}$. Hence, $\Delta f(y) = f(y)$.

Verification of continuity of Δ is left to the reader. $\square$

Corollary 4.4.2. *If a weakly normal functor F generates a monad, a compact group G acts on X, and there exists a G-right-invariant $\mathcal{M} \in FG$, and FX is an absolute retract, then FX is a G-absolute retract.*

Corollary 4.4.3. *Let G be a compact group. Then PX is a G-*AR *whenever X is a compact metrizable space;* $\exp X$ *is a G-*AR *whenever X is a Peano-continuum; $GX, NX, \lambda X$ a G-*AR *whenever X is a continuum.*

Proof. We cannot use immediately the previous result only for the superextension functor. Now we consider this case. In spite of the fact that a G-invariant point in λG could not exist, nevertheless we can construct an "averaging" operator for λ. We need the following G-retraction $r \colon NX \to \lambda X$ (see M. Zarichnyi [1991a]). Let μ be a G-invariant measure with $\mu(U) > 0$ for each non-void open set $U \subset X$. Set $\mathcal{M}_0 = \{A \in \exp X \mid \mu(A) \geq 1/2\}$. Then $\mathcal{M}_0$ is a G-fixed point in λX. Let $r \colon NX \to \lambda X$ be defined by the formula

$$r(\mathcal{A}) = \mathcal{A} \cup \{M \in \mathcal{M}_0 \mid M \cap A \neq \varnothing \text{ for each } A \in \mathcal{A}\}, \quad \mathcal{A} \in NX.$$

For every $f\colon Y \to \lambda X$ consider the map $\Delta_\lambda f\colon Y \to \lambda X$, $\Delta_\lambda f(y) = r \circ \Delta_N f(y)$, where Δ_N is an averaging operator for the functor N determined with a G-right-invariant point $\mathcal{M}_N = \{\{G\}\} \in NG$. There is an obvious counterpart of Proposition 4.4.1 for this operator. □

Theorem 4.4.4. *Let X be a non-degenerate metrizable G-continuum for a compact group G, and H a subgroup of G. Then the spaces of fixed points $NX[H]$ and $\lambda X[H]$ with respect to the subgroup H are homeomorphic to the Hilbert cube.*

Proof. We consider only the case of $\lambda X[G]$. First, $\lambda X[G] \in \mathrm{AR}$. Let $\mu \in PX$ be the invariant measure with the property that $\mu(U) > 0$ for every nonempty open subset U of X.

Let now G act transitively on X. Then $\mu(\{x\}) = 0$ for each $x \in X$. There exists a closed nowhere dense subset $A \subset X$ such that $\mu(A) \geq 1/2$. Without restriction of generality we can suppose that $d(A, X) < \varepsilon$ for some invariant metric d on X. Then $\eta = C \supset \{d(X, A \cup gA) \mid g \in G\} > 0$. Put $B = X \setminus O_{\eta/2}(A)$, then $\mathcal{A} = \{C \in \exp X \mid C \supset gA$ for some $g \in G\} \in NX$ and $\mathcal{B} = \{C \in \exp X \mid C \supset gB$ for some $g \in G\} \in NX$. The maps $r_1, r_2\colon \lambda X[G] \to \lambda X[G]$, $r_1(\mathcal{M}) = \xi(\mathcal{M}, \mathcal{A})$, $r_2(\mathcal{M}) = \xi(\mathcal{M}, \mathcal{B})$ are well-defined ε-close maps with disjoint images. By the Toruńczyk characterization theorem for Q-manifolds, $\lambda X[G] \cong Q$.

If G fails to act transitively on X, we can choose two closed invariant subsets $A, B \in \exp X$ such that $d(A, X) < \varepsilon$, $d(B, X) < \varepsilon$ and $A \cap B = \varnothing$. Put $\mathcal{A} = \{C \in \exp X \mid C \supset A\}$, $\mathcal{B} = \{C \in \exp X \mid C \supset B\}$ and proceed as above. □

4.4.2. Equivariant Hilbert cubes produced by functors

A model space of the equivariant theory of Q-manifolds in the category of G-spaces is the *equivariant Hilbert cube* $\mathbf{Q}$ which was defined by M. Steinberg and J. West as the countable power of the product of the unit balls of all irreducible orthogonal real representations of G considered with the diagonal action of G. Below we represent G-equivariant Hilbert cubes for finite groups G as hyperspaces and probability measure spaces.

Let us recall some notions and facts. A G-space Z satisfies the *G-disjoint approximation property* (G-DAP) if for any $\varepsilon > 0$ there exist

G-maps $f_1, f_2\colon Z \to Z$ with $f_1(Z) \cap f_2(Z) = \varnothing$ which are ε-close to the identity id_Z.

The following result is a reformulation of the characterization of equivariant Hilbert cubes (S. Ageev [1994]).

Theorem 4.4.5. *Let a G-space Z be G-AR. Then Z and* $\mathbf{Q}$ *are equimorphic iff the following conditions hold:*

1) Z has G-DAP;

2) there exists a G-invariant point in Z;

3) let H be a closed proper subgroup of G; then any H-invariant point in Z can be approximated by points with the stabilizer H.

Definition 4.4.6. An action of a group G on X is called *strongly effective* if for every closed proper subgroup H of G and $g \in G \setminus H$ there exists $x \in X$ with $gx \notin Hx$.

If an action of G on X is such that for some $x \in X$ we have $G_x = \{e\}$, then this action is strongly effective. In particular, any free action (i.e., such that $G_x = \{e\}$ for all $x \in X$) is strongly effective.

Theorem 4.4.7. *Let G be a finite group and X a compact metrizable G-space. Then PX is G-*$\mathbf{Q}$*-equivariant Hilbert cube iff X is infinite and the action of G on X is strongly effective.*

Proof. Necessity. The X must be infinite for $PX \cong Q$. Suppose that the action of G on X is not strongly effective. Consider a proper subset $H \subset G$ and an element $g \notin H$ such that $gx \in Hx$ for every $x \in X$.

Let μ be an arbitrary H-invariant measure. Show that μ is also g-invariant. Indeed, suppose that $H = \{h_1, \dots, h_n\}$, and consider an arbitrary Borel set M in X. Put

$$N_1 = \{x \in gM \mid x = gy = h_1 y \text{ for some } y \in M\}, \tag{4.3}$$

$$N_i = \{x \in gM \setminus \textstyle\bigcup_{j<i} N_j \mid x = gy = h_i y \text{ where } y \in M\},\ i = 2, \dots, n, \tag{4.4}$$

$$M_i = g^{-1} N_i, \quad i = 1, \dots, n. \tag{4.5}$$

Then N_i, M_i are Borel sets, $M = \bigsqcup_{i=1}^n M_i$, and

$$\mu(gM) = \mu\Big(\bigsqcup_{i=1}^n N_i\Big) = \sum_{i=1}^n \mu(N_i) = \sum_{i=1}^n \mu(gM_i) = \tag{4.6}$$

$$= \sum_{i=1}^n \mu(h_i M_i) = \sum_{i=1}^n \mu(M_i) = \mu(M). \tag{4.7}$$

Hence, PX does not contain a point with the stabilizer H. But this contradicts to Theorem 4.4.5.

Sufficiency. By Corollary 4.4.3 PX is G-AR. Moreover, to obtain this one had to recall that there exits a G-invariant measure on X, namely, the Haar measure.

Show that PX possesses the G-DAP. Since X is infinite and G finite, there exists a non-isolated orbit $G(x)$, $x \in X$, on X. We can choose a finite set $S \subset X \setminus G(x)$ and a sufficiently close to the identity map $h_1 \colon PX \to PS$. Let the map $h_2 \colon PX \to P(S \cup \{x\})$ be defined by the formula $h_2 = \frac{n-1}{n} h_1 + \frac{1}{n}\delta_x$, where n is such that the map h_2 is sufficiently close to the identity. Now we can "average" the map h_1 and h_2 to equivariant maps $\Delta h_1 \colon PX \to P(G \cdot S)$ and $\Delta h_2 \colon PX \to P(G \cdot S \cup G(x))$ (here Δ is the determined by Proposition 4.4.1 averaging operator). For this maps we obtain

$$x \in \operatorname{supp} \Delta h_2(\mu_1) \setminus \operatorname{supp} \Delta h_1(\mu_2)$$

for all $\mu_1, \mu_2 \in PX$. So $\Delta h_1(PX) \cap \Delta h_2(PX) = \varnothing$ and $\Delta h_1, \Delta h_2$ can be arbitrarily close to the identity. Hence, we have the G-disjoint approximation property for PX.

Now we have only to obtain condition 3) of Theorem 4.4.5.

Let H be a proper subgroup of G, μ an H-invariant measure on X. We desire to approximate μ by measures with the stabilizer H. Fix any $\varepsilon > 0$.

Let $G \backslash H = \{g_1, \ldots, g_n\}$ and $x_i \in X$ be a point such that $g_i x_i \notin H x_i$, $i = 1, \ldots, n$.

Note that there exists an H-invariant $\frac{\varepsilon}{2}$-close to μ measure μ' with support $\bigcup_{i=1}^m H x_i$, $m \geq n$. Indeed, since the space of measures with finite supports is dense in PX, one can construct a sufficiently close to μ measure of the form $\sum_{i=1}^m \alpha_i x_i$. Now using an H-invariant on Hx_i measure λ_i, one only has to put $\mu' = \sum_{i=1}^m \alpha_i \lambda_i$.

Let $\delta > 0$ be such that, changing weights of points hx_i, $h \in H$, $i = 1, \ldots, n$, of the measure μ' by values less than δ, one obtains an ε-close to μ measure.

Since $g_1 x_1 \notin Hx_1$, changing weights of the points hx_1 and $g_1 hx_1$, $h \in H$ of the measure μ' by values less than $\frac{\delta}{2}$, we can construct an H-invariant measure μ_1 with $\mu_1(g_1 Hx_1) \neq \mu_1(Hx_1)$. Put

$$\varepsilon_1 = \min\{\frac{\delta}{4}, \frac{1}{2|Hx_1|}|\mu_1(g_1 Hx_1) - \mu_1(Hx_1)|\}.$$

Let $1 < k < n$. Suppose that there exists an H-invariant measure μ_k, satisfying $\mu_k(g_i Hx_i) \neq \mu_k(Hx_i)$, $i \leq k$, and

$$|\mu_k(\delta_z) - \mu'(\delta_z)| < \frac{2^k - 1}{2^k}\delta$$

for every point $z \in H \cdot \{x_1, \ldots, x_n\}$. Set

$$\varepsilon_k = \min\{\frac{\delta}{2^{k+1}}, \min_{1 \leq i \leq k}\{\frac{1}{2|Hx_i|}|\mu_k(g_i Hx_i) - \mu_k(Hx_i)|\}\}.$$

Since $g_{k+1}x_{k+1} \notin Hx_{k+1}$, changing weights of the points hx_{k+1} and $g_{k+1}hx_{k+1}$, $h \in H$, of the measure μ_k by values less than ε_k, we can construct an H-invariant measure μ_{k+1} such that

$$\mu_{k+1}(g_i Hx_i) \neq \mu_{k+1}(Hx_i), \qquad i \leq k+1,$$

and continue the inductive procedure.

Finally, we obtain that the measure μ_n is H-invariant, ε-close to μ, and has the stabiliser H. □

Definition 4.4.8. An action of G on X is called *strongly locally effective* if for each proper closed subgroup H of G, $g \notin H$, and an open set $U \subset X$ there exists $x \in HU$ with $gHx \neq Hx$.

Theorem 4.4.9. *Let G be a finite group and X a compact metrizable G-space. Then $\exp X$ is equimorphic to G-equivariant Hilbert cube iff X is a non-degenerate Peano continuum and the action of G on X is strongly locally effective.*

Proof. Necessity. By D. Curtis and R. Schori [1978], X have be a non-degenerate Peano continuum for the equality $\exp X \cong Q$. Now suppose that the action of G on X is not strongly locally effective. Then there exist a closed subgroup H of G, an element $g \notin H$ and an open set $U \subset X$ such that for all $x \in HU$ we have $gHx = Hx$. For some $t \in U$ consider the H-invariant point $A = Ht \in \exp X$. The open in $\exp X$ set $W = \{C \in \exp X \mid C \subset HU\}$ is a neighborhood of A. Let $B \in W$ be H-invariant. For $b \in B$ we have $b \in HU$, therefore $gb \in gHb = Hb \subset B$. Hence, $gB \subset B$. Moreover, $b \in Hb = gHb \subset gB$. Thus $gB = B$ and the stabilizer of B does not equal H. Hence, G fails to satisfy condition 3) of Theorem 4.4.5 (because the functor exp has continuous supports).

Sufficiency. By Corollary 4.4.3 $\exp X$ is G-AR.

Verify the G-disjoint approximation property for $\exp X$. Since X is the Peano continuum, there exists a convex metric on X, and therefore we can approximate the identity $\mathrm{id}_{\exp X}$ by maps of the form

$$f_1\colon \exp X \to \exp X \setminus \exp_\omega X$$

(see R. Bing [1952]). Moreover, we can also approximate the map $\mathrm{id}_{\exp X}$ by maps of the form $f_2\colon \exp X \to \exp_\omega X$ (see D. Curtis and R. Schori [1978]). Average f_1 and f_2 to Δf_1 and Δf_2, respectively (where $\Delta f_i = \bigcup_{g\in G} g^{-1} f_i g$, $i = 1, 2$). We obtain that $\Delta f_1(\exp X) \subset \exp X \setminus \exp_\omega X$ and $\Delta f_2(\exp X) \subset \exp_\omega X$. Therefore, $\Delta f_1(\exp X) \cap \Delta f_2(\exp X) = \varnothing$, and moreover, Δf_1 and Δf_2 approximate the identity. Hence, $\exp X$ satisfies the G-DAP.

We have only to prove condition 3) of Theorem 4.4.5. Let A be an H-invariant point in $\exp X$. There exist $t_1, \dots, t_n \in X$ such that $A' = \bigcup_{i=1}^n Ht_i$ is arbitrarily close to A and H-invariant. Let (U_n) be a sequence of open balls in $X \setminus (Gt_1 \cup \dots \cup Gt_n)$ such that $\operatorname{diam} U_n < 1/2^n$; $GU_i \cap GU_j = \varnothing$, $i \neq j$; and the sequence of their centers converges to t_1. Let $F_n = \{g \in G \mid d(g, H) \geq 1/n\}$. For each $g \in F_n$ there exists $u(g) \in HU_n$ with $gHu(g) \neq Hu(g)$. There exist neighborhoods $V_g \subset G$ of g and $W_g \subset GU_n$ of $u(g)$ such that $g'Hu' \neq Hu'$ for all $g' \in V_g$, $u' \in W_g$. Let $\{V_{g_1}, \dots, V_{g_m}\}$ be a cover of F_n. Fix points $u_i \in W_{g_i}$ such that $Gu_i \cap Gu_j = \varnothing$, $i \neq j$. Consider $T_n = \bigcup_{i=1}^m Hu_i$. Then for all $g \in F_n$ we have $gT_n \neq T_n$.

Since $G \setminus H = \bigcup_{n=1}^\infty F_n$, for the closed H-invariant set $A'' = A' \cup \bigcup_{n=1}^\infty T_n$ we obtain that $gA'' \neq A''$ for all $g \notin H$. Hence, $G_{A''} = H$.

Without restriction of generality, we can also suppose that A'' is arbitrarily close to A. □

4.5. Shape and homotopy properties of functors

4.5.1. Functors and n-homotopy relation

Definition 4.5.1. Two maps $f_1, f_2\colon X \to Y$ are called *n-homotopic* (this is denoted by $f_1 \overset{n}{\sim} f_2$) if for every paracompact Hausdorff Z, $\dim Z \le n$, and maps $g\colon Z \to X$ maps $f_1 \circ g$ and $f_2 \circ g$ are homotopic.

Lemma 4.5.2. **(Chigogidze lemma)** *Maps $f_1, f_2\colon X \to Y$ are n-homotopic iff there exist a compact metrizable space B and an n-invertible map $g\colon B \to X$ such that $f_1 \circ g \sim f_2 \circ g$.*

Proposition 4.5.3. *Let F be a normal functor with finite supports. Then F preserves the n-homotopy relation.*

Proof. Let $f_1, f_2\colon X \to Y$ be n-homotopic maps. By Lemma 4.5.2 there exists an n-invertible maps $g\colon B \to X$ such that $f_1 \circ g \sim f_2 \circ g$. Thus, $Ff_1 \circ Fg \sim Ff_2 \circ Fg$ and by Theorem 4.3.1 the map Fg is n-invertible. Finally, by Lemma 4.5.2, $Ff_1 \overset{n}{\sim} Ff_2$. □

Definition 4.5.4. Suppose that a compact metrizable space X is a subset in the Hilbert space l_2. The space X is said to be *n-movable* if for every neighborhood U of X in H there exists a neighborhood U_0 of X in H satisfying the property: for every metrizable compact space $A \subset U_0$ with $\dim A \le n$ and neighborhood $\widehat{U}$ of X there exists a homotopy $\varphi\colon A \times [0,1] \to U$ such that $\varphi(x,0) = x$, $\varphi(x,1) \in \widehat{U}$, $x \in X$.

Recall that ANR-system $\mathbb{X} = \{X_\alpha, p\alpha'\alpha\}$ is called *n-movable* if for every α there exists $\alpha' \ge \alpha$ satisfying the following: for every $\alpha'' \ge \alpha'$ and $f'\colon P \to X_{\alpha'}$, where P is a finite polyhedron with $\dim P \le n$, there exists a map $f''\colon P \to X_{\alpha''}$ such that $p_{\alpha'\alpha} \circ f' \sim p_{\alpha''\alpha} \circ f''$. It turns out that a compact metrizable space X is n-movable iff it has an n-movable associated ANR-system (see Kozlowski, Segal [1974]). By an ANR-sequence we understand an inverse system $\mathcal{S} = \{X_i, p_i^j\}$ over positive integers such that $X_i \in \mathrm{ANR}$ for every i.

Theorem 4.5.5. *Let F be a normal functor with finite supports that preserves the class of compact metrizable* ANR*-spaces. Then F preserves the n-movability of compact metrizable spaces.*

Proof. Let X be an n-movable metrizable compact space and $\mathbb{X} = (X_\alpha, p_{\alpha\alpha'})$ an n-movable associated ANR-system. Show that an ANR-system $F\mathbb{X} = (FX_\alpha, Fp_{\alpha\alpha'})$ is also n-movable.

Choose for α an index $\alpha' \geq \alpha$ from the definition of n-movable ANR-system. It turns out that this α' fits also for an ANR-system $F\mathbb{X}$. Show this.

Let $f'\colon P \to FX_{\alpha'}$ be an F-valued map of a finite polyhedron P, $\dim P \leq n$, $\Gamma_{f'}$ a graph of this map, i.e.

$$\Gamma_{f'} = \{(x,y) \mid y \in \operatorname{supp}(f(x))\} \subset P \times X_{\alpha'}.$$

Since $\dim \Gamma_{f'} \leq n$ and $X'_\alpha \in$ ANR, there exist maps $g'\colon \Gamma_{f'} \to L$, $h'\colon L \to X_{\alpha'}$, where L is a finite polyhedron with $\dim L \leq n$, such that $\mathrm{pr}_2 |\Gamma_{f'} \sim h' \circ g'$.

Let $\alpha'' \leq \alpha'$. Then there exists a map $h''\colon L \to X_{\alpha''}$ such that $p_{\alpha'\alpha} \circ h' \sim p_{\alpha''\alpha} \circ h''$. Define a map $k\colon P \to F\Gamma_{f'}$ by the formula: $k(x) = Fi_x \circ f'(x)$, $x \in P$ (here we denotes by i_x the following map $i_x(y) = (x,y)\colon X_{al'} \to P \times X_{\alpha'}$). Put $f'' = F(h'' \circ g') \circ k$. It is easy to show that f'' is continuous. Moreover,

$$\begin{aligned} Fp_{\alpha''\alpha} \circ f'' = F(p_{\alpha''\alpha} \circ h'' \circ g') \circ k \sim F(p_{\alpha'\alpha} \circ h' \circ g') \circ k \sim \\ \sim Fp_{\alpha'\alpha} \circ F(\mathrm{pr}_2 |\Gamma_{f'}) \circ k = Fp_{\alpha'\alpha} \circ f'. \end{aligned}$$

Hence, every ANR-system $F\mathbb{X}$ is n-movable. Therefore FX is an n-movable space. □

4.5.2. Functors and (simple) connectedness

Proposition 4.5.6. *Let F be a continuous, monomorphic, preserving intersections, singletons and empty set functor. If X is a connected space, then so is FX.*

Proof. We first prove this for the functors of finite degree. Suppose that $\deg(F) = n$. The natural map $\pi\colon X^n \times Fn \to FX$ can be factorized through the quotient map

$$X^n \times Fn \to (X^n \times Fn)/(\{(x_0, \ldots, x_0)\} \times Fn),$$

where x_0 is an arbitrary point of X (this is an easy consequence of the preservation of the singletons). Since the latter space is connected, so is FX.

The general case follows from the fact that $FX = \overline{\bigcup_{i=1}^\infty F_i X}$. □

Theorem 4.5.7. *Let F be a continuous, monomorphic, preserving intersections and empty set functor of finite degree n, $F(n) \in$ ANR, and $F1$ is arcwise connected. Then the following conditions are equivalent:*

1) for every compact metrizable space X with the homotopy type of finite connected polyhedron a space FX is simply connected;

2) for every compact connected ANR-space X the space FX is simply connected;

3) FS^1 is simply connected;

4) every map $\alpha\colon S^1 \to FS^1$ with $\alpha(S^1) \subset F_2S^1$ is homotopic to the map ηS^1.

Proof. Since every compact metrizable ANR-space has the homotopy type of finite polyhedron (see Chapman[1976]) and every continuous functor preserves the homotopy equivalence of spaces, we have 1) $\Longleftrightarrow$ 2).

2)$\Rightarrow$3)$\Rightarrow$4) are obvious.

Hence we have to show only 4)$\Rightarrow$1). For this we prove six claims. Below X is a finite polyhedron, and a functor F satisfies 4).

Claim 4.5.8. *For every nonempty finite space k a space F_kX is arcwise connected.*

Proof. Fix $x \in X$. Every point of F_kX is of form $F\xi(r)$ for some $\xi \in C(k, X)$, $r \in Fk$. Since X is arcwise connected, we can connect points ξ and η in $C(k, X)$, where η is a constant map of k to the point x. Then we can connect points $F\xi(r)$ and $F\eta(r) \in F\{x\}$ in F_kX. Since $F\{x\}$ is arcwise connected, F_kX is also such a space. □

Claim 4.5.9. *Let $k \geq 2$ be such that $F_kX \neq F_{k-1}X$. For every map $\alpha\colon S^1 \to FX$ with $\alpha(S^1) \subset F_kX$ there exist a homotopic to α map $\omega\colon S^1 \to FX$, a set $W \subset S^1$ representing as a finite union of distinct closed intervals, and a map $\psi\colon W \to C(k, X) \times Fk$ such that*

$$\pi_{F,k}X \circ \psi = \omega|W \quad \text{and} \quad \omega^{-1}(F_kX \setminus F_{k-1}X) \subset W,$$

where $\pi_X : X^k \times Fk \to FX$ is the Basmanov map.

Proof. Since F_kX is arcwise connected, there exists a homotopic to α map $\beta\colon S^1 \to FX$ such that a set $V = \beta^{-1}(F_kX \setminus F_{k-1}X)$ is a proper subset of S^1. The set V is an at most than countable union of mutually disjoint intervals. Moreover, a restriction of $\pi_{F,k}X$ onto $\pi_{F,k}X^{-1}(F_kX \setminus$

$F_{k-1}X$) is a locally trivial fibration. Therefore, there exists a map $\varphi\colon V \to C(k, X) \times Fk$ with $\pi_{F,k}X \circ \varphi = \beta|V$. Since

$$M = \pi_{F,k}X^{-1}(F_{k-1}X) \in \mathrm{ANR},$$

there exists a homotopy $h_t\colon C(k, X) \times Fk \to C(k, X) \times Fk$ satisfying the following conditions: $h_0 = \mathrm{id}$, h_1 retracts some neighborhood of M onto M, and $h_t(z) = z$ for every $t \in [0, 1]$, $z \in M$ (see K. Borsuk [1967]). Define a homotopy $\gamma_t\colon S^1 \to FX$ by the following properties: $\gamma_t|S^1\setminus V = \beta|S^1\setminus V$, $\gamma_t|V = \pi_{F,k}X \circ h_t \circ \varphi$. This homotopy is continuous and $\gamma_0 = \beta$. Put $\omega = \gamma_1$. Choose a finite union W of mutually disjoint intervals such that $\overline{\omega^{-1}(F_kX \setminus F_{k-1}X)} \subset W \subset V$. Finally, set $\psi = h_1 \circ \varphi|W$. □

Claim 4.5.10. *Let $k \geq 3$ and maps*

$$\alpha\colon [0, 1] \to FX, \quad \psi\colon [0, 1] \to C(k, X) \times Fk$$

be such that $\alpha = \pi_{F,k}X \circ \psi$, $\alpha(\{0, 1\}) \subset F_{k-1}X$. Then there exists a map $\omega\colon [0, 1] \to FX$ with $\omega \sim \alpha$ $\mathrm{rel}\{0, 1\}$ and $\omega([0, 1]) \subset F_{k-1}X$.

Proof. Fix a point $x \in X$. By arcwise connectedness of X there exists a map $\varphi\colon [0, 1] \to C(k, X) \times Fk$ such that $\varphi \sim \psi$ $\mathrm{rel}\{0, 1\}$, $\pi_{F,k}X \circ \varphi([0, 1/4]\cup[3/4, 1]) \subset F_{k-1}X$, and $\mathrm{pr}_1 \circ\varphi(1/4)$ and $\mathrm{pr}_1 \circ\varphi(3/4)$ coincide with $k \to \{x\} \in X$. Let $\varrho\colon [0, 1] \to C(k, X) \times Fk$ be a map such that $\varrho \sim \varphi$ $\mathrm{rel}[0, 1/4] \cup [3/4, 1]$ and $\left|\{m \in k \mid \mathrm{pr}_1 \circ\varrho(a)(m) \neq x\}\right| \leq 1$ for every point $a \in [1/4, 3/4]$. Put $\omega = \pi_{F,k}X \circ \varrho$. Then $\omega \sim \alpha$ $\mathrm{rel}\{0, 1\}$ and $\omega([0, 1]) \subset F_{k-1}X$, because $\omega([1/4, 3/4]) \subset F_2X \subset F_{k-1}X$. □

Claim 4.5.11. *Let a map $\alpha\colon [0, 1] \to FX$ be such that $\alpha([0, 1]) \subset F_2X$, $\alpha(0) \in F_1X$, and $\alpha = \pi_{F,2}X \circ \psi$ for some continuous map $\psi\colon [0, 1] \to C(2, X) \times F2$. Then there exists a map $\omega\colon [0, 1] \to FX$, such that $\omega \sim \alpha$ $\mathrm{rel}\{0, 1\}$, $\omega([0, 1/4]) \subset F_1X$, and $\omega = \pi_{F,2}X \circ \varphi$ for some map $\varphi\colon [0, 1] \to C(2, X) \times F2$ satisfying the following: $\mathrm{pr}_1 \circ\varphi(1/4)\colon 2 \to X$ is constant.*

Proof. Since $\pi_{F,2}X \circ \psi(0) \in F_1X$, either $\mathrm{pr}_1 \circ\psi(0)$ is a constant map $2 \to X$ or $\mathrm{pr}_2 \circ\psi(0) \in F_12$. Let $\gamma\colon [0, 1] \to [0, 1]$ be a homotopic to $1_{[0,1]}$ map such that $\gamma([0, 1/2]) = \{0\}$, $\gamma(1) = 1$. If $\mathrm{pr}_1 \circ\psi(0)$ is constant, put $\varphi = \psi \circ \gamma$, $\omega = \alpha \circ \gamma$. Otherwise, let $\mathrm{pr}_2 \circ\psi(0) \in F\{0\}$. Since X is

arcwise connected, one can construct a map $\varphi\colon [0,1] \to C(2,X) \times F2$ satisfying the following conditions:

$$\begin{aligned}
&\varphi \sim \psi \circ \gamma \ \operatorname{rel}\{0\} \cup [1/2, 1],\\
&\operatorname{pr}_1 \circ \varphi(t)(0) = \operatorname{pr}_1 \circ \psi \circ \gamma(t)(0) \quad t \in [0,1],\\
&\operatorname{pr}_2 \circ \varphi = \operatorname{pr}_2 \circ \psi \circ \gamma\\
&\operatorname{pr}_1 \circ \varphi(1/4)(0) = \operatorname{pr}_1 \circ \varphi(1/4)(1).
\end{aligned}$$

Finally set $\omega = \alpha \circ \gamma$. $\square$

Claim 4.5.12. *Let $\alpha\colon [0,1] \to FX$ be a map such that $\alpha([0,1]) \subset F_2X$, $\alpha(\{0,1\}) \subset F_1X$, and $\alpha = \pi_{F,2}X \circ \psi$ for some $\psi\colon [0,1] \to C(2,X) \times F2$. Moreover, let $\operatorname{pr}_1 \circ \psi(0)$ and $\operatorname{pr}_1 \circ \psi(1)$ be constant maps. Then there exists a map $\omega\colon [0,1] \to FX$ with $\omega \sim \alpha \ \operatorname{rel}\{0,1\}$ and $\omega([0,1]) \subset F_1X$.*

Proof. Represent S^1 as a quotient space of $\{0,1\} \times [0,1]$ under an equivalence relation whose non-trivial elements are only $\{0,1\} \times \{0\}$ and $\{0,1\} \times \{1\}$. Let g be a corresponding quotient map. Define maps $\lambda\colon S^1 \to X$, $\delta\colon [0,1] \to C(2,S^1)$ by the formulae:

$$\lambda\big(g(j,t)\big) = \operatorname{pr}_1 \circ \psi(t)(j), \quad \delta(t)(j) = g(j,t) \quad t \in [0,1],\ j \in \{0,1\}.$$

Consider the following map $\mu\colon [0,1] \to FS^1$, $\mu(t) = F\big(\delta(t)\big)\big(\operatorname{pr}_2 \circ \psi(t)\big)$ for $t \in [0,1]$. Since $\lambda \circ \delta(t)(j) = \operatorname{pr}_1 \circ \psi(t)(j)$ for all $t \in [0,1]$, $j \in \{0,1\}$, we have $\alpha = F\lambda \circ \mu$. By $\mu(\{0,1\}) \subset F_1S^1$ there exists a map $\mu^1\colon [0,1] \to FS^1$ such that $\mu^1([0,1]) \subset F_1S^1$ and $\mu^1|\{0,1\} = \mu|\{0,1\}$. Condition 4) (of the theorem) implies that $\mu^1 \sim \mu \ \operatorname{rel}\{0,1\}$. Now we have only to put $\omega = F\lambda \circ \mu^1$. $\square$

Claim 4.5.13. *Let $\alpha\colon S^1 \to FX$ be a map such that $\alpha(S^1) \subset F_1X$. Then α is homotopic to a constant map.*

Proof. Since $\pi_{F,1}X$ is homeomorphic to a surjective map $C(1,X) \times F1 \to F_1X$, we have $\alpha = \pi_{F,1}X \circ \psi$ for some $\psi\colon [0,1] \to C(1,X) \times F1$. Define maps $\tau\colon S^1 \to X$ and $\lambda\colon S^1 \to C(1,S^1)$ by the following formulae:

$$\tau(s) = \operatorname{pr}_1 \circ \psi(s)(0), \quad \lambda(s)(0) = s \quad s \in S^1.$$

Consider a map $\mu\colon S^1 \to FS^1$, $\mu(s) = F\big(\lambda(s)\big)\big(\operatorname{pr}_2 \circ \psi(s)\big)$ for $s \in S^1$. Then $\alpha = F\tau \circ \mu$. Thus, by condition 4) the map α is homotopic to a constant map. $\square$

Hence, we proved 4)⇒1) and the theorem. □

Corollary 4.5.14. *The functors $\lambda_n, \exp_n, P_n$, $n \geq 3$, transform connected compact ANR-spaces to simply connected spaces.*

Exercises

1. Show that there is no weakly normal functor of finite degree such that $F(2^\omega)$ is a locally connected space.
2. Let F be a weakly normal functor of finite degree that preserves the class of compact metrizable ANR-spaces. Show that FS^1 is not contractible. Is it possible to remove here the condition of preserving ANRs?

4.6. Notes and comments to Chapter 4

Theorem 4.2.3 is proved by V. Basmanov [1983]. Note that the second part of the proof (reduction of general case to the case of finite-dimensional spaces) literally follows the arguments by V. Fedorchuk [1981]. The Basmanov theorem generalizes the series of results in this direction; see V. Fedorchuk [1981]. Theorem 4.2.12 is due to A. Savchenko [1989].

The definition of n-homotopy follows A. Chigogidze [1987]. This paper contains also the Chigogidze lemma. Theorem 4.2.13 generalizes Corollary 4.2.14 by A. Dranishnikov [1991].

For Proposition 4.4.1 and Corollary 4.4.3 see A. Teleiko [1997].

Theorem 4.4.4 is proved by M. Zarichnyi [1991a].

Theorem 4.5.7 and Corollary 4.5.14 are due to V. Basmanov [1984].

Chapter 5.

Functors and nonmetrizable compacta

This chapter provides results whose main feature is the "effect of uncountability" discovered by E. Shchepin. Section 5.1 provides a method of investigations of nonmetrizable functor-powers, i. e. the spaces of the form $F(K^\tau)$, where K is a compact metrizable space and $\tau > \omega$.

In Section 5.3 we characterize the class of openly generated continua (compact Haussdorff spaces) in terms of (0-)softness of the multiplication maps of the superextension monad and the inclusion hyperspace monad.

Section 5.3 contains also a characterization theorem for the power monad in the class of normal monads in terms of softness of the multiplication map (see subsection 5.3.3).

5.1. Homeomorphisms of uncountable functor-powers

In this section we consider the problem of invariance of degree under homeomorphisms of the functor-powers, i. e. the spaces of the form $F(K^\tau)$, where $K \in |M\mathbf{Comp}|$ and $\tau > \omega_1$.

5.1.1. Method of characteristics

Let $f\colon X \to Y$ be a map. A point $y \in Y$ is called a *splitting point* of f if there exist open sets U, V in Y such that $y \in \bar{U} \cap \bar{V}$ and $\overline{f^{-1}(U)} \cap$

$\overline{f^{-1}(V)} = \varnothing$.

Lemma 5.1.1. *Suppose that a map f of compact Hausdorff spaces admits a decomposition, $f = gh$, where h is a coretraction and g an irreducible map. Then the set of irreducible points of the map f coincides with the set of multiple points of g.*

Proof. Suppose that y is a multiple point of $g\colon Y \to Z$. Then $y = g(x_1) = g(x_2)$. Let U_1, U_2 be neighborhoods of respectively x_1 and x_2 such that $\overline{U_1} \cap \overline{U_2} = \varnothing$. Put $V_i = g^\sharp(U) = \{z \in Z \mid g^{-1}(z) \subset U_i\}$ (the *small image* of U_i. Since g is irreducible, we have $y \in \overline{V_i}$, $i = 1, 2$. Besides, V_i are open, because g is a closed map. Obviously, $\overline{f^{-1}(V_1)} \cap \overline{f^{-1}(V_2)} = \varnothing$, i. e. y is a splitting point of f.

Conversely, suppose that y is a nonmultiple point of g, $x = g^{-1}(y)$, and $y \in \overline{V_1} \cap \overline{V_2}$. Then $x \in \overline{g^{-1}(V_1)}$, otherwise $g^\sharp(Y \setminus \overline{g^{-1}(V_1)}$ would be a nonempty neighborhood of y that misses V_1, a contradiction. Similarly, $x \in \overline{g^{-1}(V_2)}$. Denote by s a right inverse to f map. Then

$$s(x) = \overline{s(g^{-1}(V_1))} \cap \overline{s(g^{-1}(V_2))} \subset \overline{f^{-1}(V_1)} \cap \overline{f^{-1}(V_2)}$$

and we have proved that y is not a splitting point of f. □

Lemma 5.1.2. *Let K be a compact metric space and A, B disjoint countable sets. There exists a sequence of maps $g_n\colon K^{A \cup B} \to K^{A \cup B}$ such that the following holds:*

1) g_n converges to $1_{K^{A \cup B}}$;

2) all compositions πg_n, where $\pi\colon K^{A \cup B} \to K^B$ is the projection, are embeddings.

Proof. We may suppose that A is the set of all odd and B the set of all even natural numbers. Fix a point $* \in K$ and define the map $g_n\colon K^{\mathbb{N}} \to K^{\mathbb{N}}$ by the formula:

$$\begin{aligned} g_n(x_1, x_2, x_3, \dots, x_n, \dots) =& (x_1, x_2, x_3, \dots, x_{2n}, *, x_1, , *, x_3, *, \dots, \\ & *, x_{2n-1}, *, x_{2n+1}, *, x_{2n+2}, *, x_{2n+3}, *, \dots), \end{aligned}$$

where $(x_1, x_2, x_3, \dots, x_n, \dots) \in K^{\mathbb{N}}$. Obviously, we obtain a required sequence. □

Recall that a *morphism* of a diagram $\pi^2 K_1$ to a diagram $\pi^2 K_2$ is a quadruple of maps f_1, f_{12}, f_{13}, and f making the diagram

$$\begin{tikzcd}
K_1^4 \arrow[r, "\pi_{13}"] \arrow[d, "\pi_{12}"'] \arrow[rr, bend left, "f"] & K_1^2 \arrow[d, "\pi_1"'] \arrow[rr, bend left, "f_{13}"] & K_2^4 \arrow[r, "p_{13}"] \arrow[d, "p_{12}"] & K_2^2 \arrow[d, "p_1"] \\
K_1^2 \arrow[r, "\pi_1"'] \arrow[rr, bend right, "f_{12}"'] & K_1 \arrow[rr, bend right, "f_1"'] & K_2^2 \arrow[r, "p_1"'] & K_2
\end{tikzcd} \tag{5.1}$$

commutative. A morphism is called an *isomorphism* if all involved maps are homeomorphisms. Similar notions are naturally defined also for the diagrams $\pi^3 K_1$ and $\pi^3 K_2$.

The following auxiliary result plays a crucial role in the method of characteristics.

Proposition 5.1.3. *Let F_1, F_2 be normal functors and K_1, K_2 two compact metric spaces homeomorphic to their countable powers. Then every isomorphism from the diagram $\pi^2 K_1$ to a diagram $\pi^2 K_2$ generates an isomorphism from the diagram $\pi^3 K_1$ to the diagram $\pi^3 K_2$.*

Proof. Suppose that the given morphism is represented by the curve arrows in the diagram (5.1). Denote by $\pi_3 \colon K_1^4 \to K_1^3$ and $p_3 \colon K_2^4 \to K_2^3$ the projections onto the first three coordinates. We are going to prove that the (multivalued) map $F_2 p_3 \circ f \circ (F_1 \pi_3)^{-1}$ is in fact single-valued.

Suppose the contrary. Then there exist $x_1, x_2 \in K_1^4$ such that $F_1 \pi_3(x_1) = F_1 \pi_3(x_2)$ but, however, $F p_3(f(x_1)) \neq F p_3(f(x_2))$. Let U_1, U_2 be neighborhoods $f(x_1)$ and $f(x_2)$ respectively such that $F p_3(U_1) \neq F p_3(U_2)$. Since K_1 is homeomorphic to K_1^ω, by Lemma 5.1.2, we can choose a sequence of maps $g_n \colon K_1^3 \to K_1^3$ converging to $1_{K_1^3}$ and such that the map $\pi_1' g_n$ is injective for every n ($\pi_1' \colon K_1^3 \to K_1$ is the projection onto the first coordinate). There exist neighborhoods V_1 and V_2 of the poitns x_1 and x_2 respectively such that $f(V_i) \subset U_i$, $i = 1, 2$. By continuity, the sequence of maps $F_1(g_n \times 1_{K_1})$ converges to the map $1_{F(K_1^4)}$ and there exists $n \in \mathbb{N}$ such that

$$x_i' = F_1(g_n \times 1_{K_1})(x_i) \in V_i, \ i = 1, 2.$$

Then

$$F_1\pi_3(x_1') = F_1(g_n\pi_3)(x_1) = F_1(g_n\pi_3)(x_2) = F_1\pi_3(x_2')$$

and simultaneously $Fp_3(f(x_1')) \neq Fp_3(f(x_2'))$.

Note that the maps $F_1\pi_3$ and F_2p_3 are right invertible. By Proposition 2.11.1, the maps $\chi_i = \chi F_i(\pi^2 K_i)$ and $\bar{\chi}_i = \chi F_i(\pi^3 K_i)$, $i = 1, 2$, are irreducible. The points of the image of the map $F_1(g_n)$ are the nonmultiple points of the map $\bar{\chi}_1$. By Lemma 5.1.1, the point $\chi_1(x_1') = \chi_1(x_2')$ is not the splitting point of the map χ_1, while, by the same lemma, the point $\chi_2(f(x_1')) = \chi_2(f(x_2'))$ is the splitting point of the map χ_2. We have obtained a contradiction, because, obviously, the homeomorphism of the maps χ_1 and χ_2 given by the diagram

$$\begin{tikzcd}
F_1(K_1^4) \arrow[r, "f"] \arrow[d, "\chi_1"'] & F_2(K_2^4) \arrow[d, "\chi_2"] \\
F_1(K_1^2) \times F_1(K_1^2) \arrow[r, "f_{12}\times f_{13}"] & F_2(K_2^2) \times F_2(K_2^2)
\end{tikzcd}$$

preserves the splitting points.

We have proved that the map

$$f' = F_2p_3 \circ f \circ (F_1\pi_3)^{-1} \colon F_1(K_1)^3 \to F_2(K_2)^3$$

is single-valued. The quadruple f_1, f_{12}, f_{13}, and f' determines an isomorphism of the diagrams $\pi^3 K_1$ and $\pi^3 K_2$. □

Let K be a compact metrizable space, F a normal functor. A space of the form $F(K^\tau)$ is called a *normal functor-power* of weight τ.

Theorem 5.1.4. *Suppose that two normal functor-powers $F_1(K_1\tau)$, $F_2(K_2^\tau)$ of weight $\tau > \omega_1$ are homeomorphic. Then the diagrams $F_1(\pi^3 K_1^\omega)$, $F_2(\pi^3 K_2^\omega)$ are isomorphic and, consequently, the characteristics $\chi_{K_1^\omega} F_1$, $\chi_{K_2^\omega} F_2$ coincide.*

Proof. Without loss of generality we may suppose that the spaces K_1 and K_2 are homeomorphic to their countable powers. Let $f\colon F_1(K_1^\tau) \to F_2(K_2^\tau)$ be a homeomorphism. A countable subset $A \subset \tau$ is called *admissible* with respect to f if there exists a homeomorphism $f_A\colon F_1(K_1^A) \to F_2(K_2^A)$ such that $f_A \circ F_1\pi_A = F_2p_A \circ f$, where $\pi_A\colon K_1^\tau \to K_1^A$ and

$p_A\colon K_2^\tau \to K_2^A$ are the projections. The Shchepin spectral theorem implies that every countable subset of τ is a subset of a countable admissible set. It follows from Proposition 5.1.3 that the union of two admissible sets with nonempty intersection is also amdissible. This allows us to deduce that the homeomorphism f induces an isomorphism of the diagram $F_1(\pi^3 K_1^\omega)$ and $F_2(\pi^3 K_2^\omega)$. $\square$

5.1.2. Another characterization of finite power functor

Theorem 5.1.5. *Let F be a normal functor of finite degree such that the space $F(I^\tau)$ is topologically homogeneous for some $\tau \geq \omega_2$. Then F is a power functor $(-)^n$ for some $n \in \mathbb{N}$.*

Proof. Supposing that the space $F(I^\tau)$ is topologically homogeneous for some $\tau \geq \omega_2$ choose a homeomorphism $h\colon F(I^\tau) \to F(I^\tau)$ such that, for some $a, b \in F(I^\tau)$ with $\deg(a) = n$, $\deg(b) = 1$ we have $h(a) = b$. There exists an isomorphism $H = \{h_1, h_{12}, h_{13}, h_{123})$ of the diagram $\pi^(I^\omega)$ onto itself and a point $\bar{a} \in F(I^\omega)$ such that $\deg(\bar{a}) = \deg(a) = n$ and $\deg(h_1(\bar{a})) = 1$.

Since $\deg(F) = n$, for every points $c_1, c_2 \in F(I^\omega \times I^\omega)$ such that $F\pi_1(c_1) = F\pi_1(c_2) = \bar{a}$ there exists a unique point $c \in F(I^\omega \times I^\omega \times I^\omega)$ such that $F\pi_{12}(c) = c_1$, $F\pi_{13}(c) = c_2$. In other words, the diagram

$$\begin{array}{ccc} \begin{array}{c}(F\pi_{12})^{-1}(F\pi_1)^{-1}(\bar{a}) \\ = (F\pi_{13})^{-1}(F\pi_1)^{-1}(\bar{a})\end{array} & \xrightarrow{F\pi_{13}} & (F\pi_1)^{-1}(\bar{a}) \\ {\scriptstyle F\pi_{12}}\Big\downarrow & & \Big\downarrow{\scriptstyle F\pi_1} \\ (F\pi_1)^{-1}(\bar{a}) & \xrightarrow{F\pi_1} & \{\bar{a}\} \end{array}$$

is a pullback diagram. Since H is an isomorphism of diagrams, the diagram

$$\begin{array}{ccc} (F\pi_{12})^{-1}(F\pi_1)^{-1}(h_1(\bar{a})) & \xrightarrow{F\pi_{13}} & (F\pi_1)^{-1})(h_1(\bar{a})) \\ {\scriptstyle F\pi_{12}}\Big\downarrow & & \Big\downarrow{\scriptstyle F\pi_1} \\ (F\pi_1)^{-1})(h_1(\bar{a})) & \xrightarrow{F\pi_1} & \{h_1(\bar{a})\} \end{array}$$

is also a pullback diagram. But since F preserves preimages and inter-

sections, the latter diagram is isomorphic to

$$\begin{tikzcd} F(\{h_1(\bar{a})\} \times I^\omega \times I^\omega) \arrow[r, "F\pi_{13}"] \arrow[d, "F\pi_{12}"'] & F(\{h_1(\bar{a})\} \times I^\omega) \arrow[d, "F\pi_1"] \\ F(\{h_1(\bar{a})\} \times I^\omega) \arrow[r, "F\pi_1"] & \{h_1(\bar{a})\} \end{tikzcd}$$

which means that F preserves the product of Hilbert cubes. By Lemma 2.9.2, the functor F is multiplicative and by Theorem 2.9.7 F is isomorphic to $(-)^n$. $\square$

5.1.3. Hyperspaces of powers of compact metric spaces

Definition 5.1.6. A subspace A of topological space X is called *invariant* if $h(A) = A$ for every homeomorphism $h\colon X \to X$.

Remark that any topologically homogeneous space has no proper invariant subspaces.

Theorem 5.1.7. *Let K be a Peano continuum, $n \geq 1$, and $\tau > \omega_1$. Then for every $n \in \mathbb{N}$ the hypersymmetric power $\exp_n K^\tau$ is an invariant subspace of $\exp K^\tau$.*

One can easy obtain this result as a consequence of the following Theorems 5.1.8 and 5.1.10.

Theorem 5.1.8. *Suppose that K and L are metrizable continua, $\tau > \omega_1$, and $h\colon \exp K^\tau \to \exp L^\tau$ is a homemorphism. If degrees of x and $h(x)$ are finite, then $\deg x = \deg h(x)$.*

Definition 5.1.9. Let K be a compact metrizable space, $|K| > 1$. We call K *stretchable* if there exists $\varepsilon > 0$ such that for every distinct points $x, y \in K$ the distance (with respect to a fixed metric) between $f(x)$ and $f(y)$ is $> \varepsilon$ for some map $f\colon K \to K$.

It is easy to see that both zero-dimensional compact metrizable spaces and Peano continua are stretchable.

Theorem 5.1.10. *Suppose that K and L are stretchable compact metrizable spaces, $\tau > \omega_1$, and $h\colon \exp K^\tau \to \exp L^\tau$ is a homeomorphism. Then h preserves the class of points of finite degree.*

Theorems 5.1.8 and 5.1.10 imply the following fact.

Corollary 5.1.11. *If $\tau > \omega_1$, then the space $\exp K^\tau$ is not topologically homogeneous for every metrizable compact space K, $|K| > 1$.*

Proof. By Theorem 5.1.10 $\exp K^\tau$ is not topologically homogeneous for zero-dimensional K. If K is not zero-dimensional, choose a nontrivial connected component K_0 of K. Then $\exp K_0$ is a nontrivial connected component in $\exp K^\tau$. By Theorem 5.1.8 $\exp K_0$ is not topologically homogeneous. □

The following result generalizes Theorem 5.1.7.

Corollary 5.1.12. *Let K be a stretchable metrizable continuum, $\tau > \omega_1$, and $n \in \mathbb{N}$. Then the subspace $\exp_n K^\tau$ is invariant in $\exp K^\tau$.*

Since $\exp_1 X$ and X are naturally homeomorphic, we have the following

Corollary 5.1.13. *Let K and L be strechable metrizable continua and $\tau > \omega_1$. Then $\exp K^\tau$ and $\exp L^\tau$ are homeomorphic if and only if so are K^τ and L^τ.*

Now we pass to the proofs of Theorems 5.1.8 and 5.1.10.

The following lemma is nothing but a specialization of Theorem 5.1.4.

Lemma 5.1.14. *Let K and L be compact metric spaces, $\tau > \omega_1$, and $h\colon \exp K^\tau \to \exp L^\tau$ a homeomorphism. Then for every $x \in K^\tau$ there exist a diagram isomorphism $\mathcal{H}\colon \exp \pi^3(K^\omega) \to \exp \pi^3(L^\omega)$ and a point $\overline{x} \in \exp K^\omega$ such that*

$$\deg x = \deg \overline{x}, \quad \deg h(x) = \deg h_1(\overline{x}).$$

Moreover, $\mathcal{H}$ is formed by homeomorphisms

$$\begin{aligned} h_1\colon & \exp K^\omega \to \exp L^\omega, \\ h_{12}\colon & \exp(K^\omega \times K^\omega) \to \exp(L^\omega \times L^\omega), \\ h_{13}\colon & \exp(K^\omega \times K^\omega) \to \exp(L^\omega \times L^\omega), \\ h_{123}\colon & \exp(K^\omega \times K^\omega \times K^\omega) \to \exp(L^\omega \times L^\omega \times L^\omega), \end{aligned}$$

satisfying

$$\begin{aligned} \exp \pi_{1i}(L^\omega) \circ h_{123} &= h_{1i} \circ \exp \pi_{1i}(K^\omega), \quad i = 2, 3, \\ \exp \pi_1(L^\omega) \circ h_{1i} &= h_1 \circ \exp \pi_1(K^\omega), \quad i = 2, 3. \end{aligned}$$

Definition 5.1.15. By the K-characteristic $\chi_K(x_2, x_3)$ of a couple

$$(x_2, x_3) \in \exp(K^\omega \times K^\omega) \times \exp(K^\omega \times K^\omega)$$

in the diagram $\exp \pi^3(K^\omega)$ we call the set

$$\{x \in \exp K_{123} \mid \exp \pi_{12}(x) = x_2 \text{ and } \exp \pi_{13}(x) = x_3\}.$$

Note that the isomorphisms of diagrams $\exp \pi^3(K^\omega)$ and $\exp \pi^3(L^\omega)$ preserve characteristics, i.e.

$$h_{123}(\chi_K(x_2, x_3)) = \chi_L(h_{12}(x_2), h_{13}(x_3)).$$

Remark also that the characteristic of (x_2, x_3) is nonempty if and only if $\exp \pi_1 x_2) = \exp \pi_1(x_3)$.

Lemma 5.1.16. *Suppose that a couple (x_2, x_3) is such that the element $\exp \pi_1(x_2) = \exp \pi_1(x_3)$ is of degree 1. Then $|\chi(x_2, x_3)| = 1$ iff either $\deg x_2 = 1$ or $\deg x_3 = 1$.*

Lemma 5.1.17. *Suppose that a couple (x_2, x_3) is such that the element $\exp \pi_1(x_2) = \exp \pi_1(x_3)$ is of degree 1. Then $|\chi(x_2, x_3)| = 7$ iff $\deg x_2 = \deg x_3 = 2$.*

Lemma 5.1.18. *Suppose that a couple (x_2, x_3) is such that the element $\exp \pi_1(x_2) = \exp \pi_1(x_3) = x_1$ is of degree n and $\operatorname{supp} x_1 = \{c_1, \ldots, c_n\}$. Then*

$$\chi(x_2, x_3) = \prod_{i=1}^{n} \chi(\{\pi_1^{-1} c_i \cap \operatorname{supp} x_2\}, \{\pi_1^{-1} c_i \cap \operatorname{supp} x_3\}).$$

For a point $x_1 \in \exp K_1$ define $B_{x_1}(K) \subset \exp K_{12} \times \exp K_{13}$ by the following formula

$$B_{x_1}(K) = \{(x_2, x_3) \mid \exp \pi_1(x_2) = \exp \pi_1(x_3) = x_1, |\chi(x_2, x_3)| = 7\}.$$

This definition immediately implies the following fact.

Lemma 5.1.19. *For the isomorphism $\mathcal{H}$ of $\exp \pi^3(K^\omega)$ and $\exp \pi^3(L^\omega)$ we have*

$$(h_{12} \times h_{13}) B_{x_1} = B_{h_1(x_1)}(L).$$

It turns out that the properties of $B_{x_1}(K)$ depend on the degree of x_1.

Lemma 5.1.20. *If K is a metrizable continuum and $\deg x_1 = n$, then the space $B_{x_1}(K)$ has n connected components.*

Denote by $A_n(X)$ the space $\exp_n X \setminus \exp_{n-1} X$.

Lemma 5.1.21. *If X is a continuum, then both $\exp X$ and $A_n(X)$ are connected.*

Proof of Lemma 5.1.20. Let $\operatorname{supp} x_1 = \{c_1, \dots, c_2\}$. Suppose that $(x_2, x_3) \in B_{x_1}$. Then by Lemmas 5.1.18, 5.1.16 and 5.1.17 and the definition B_{x_1} we have that for some $m \leq n$ the couple (x_2, x_3) satisfies the following:
1) $|\operatorname{supp}_i \cap \pi_1^{-1}(c_m)| = 2$, $i = 2, 3$,
2) for every $j \neq m$ and $1 \leq j \leq n$ we have either $|\operatorname{supp}_2 \cap \pi_1^{-1}(c_j)| = 1$ or $|\operatorname{supp}_3 \cap \pi_1^{-1}(c_j)| = 1$.

Moreover, it is easy to see that every pair (x_2, x_3), satisfying these properties, belongs to B_{x_1}.

For every couple $(x_2, x_3) \in B_{x_1}$ denote by $i(x_2, x_3)$ a number satisfying properties 1) and 2).

Consider the following sets $M_{x_1}^m$:

$$M_{x_1}^m = \{(x_2, x_3) \in B_{x_1} \mid i(x_2, x_3) = m\}, \quad 1 \leq m \leq n.$$

Show now that $M_{x_1}^m$ form the family of all connected components of B_{x_1}. For this, fix $l \in \{1, 2, \dots, n\}$. Let $(x_2, x_3) \in M_{x_1}^m$ and $\operatorname{supp} x_i \cap \pi_1^{-1}(c_l) = \{a_i, b_i\}$, $i = 2, 3$. Consider a base neighborhood $\langle U_i^a, U_i^b, U_i \rangle$ of the point $x_i \in \exp K_{1i}$, where U_i^a (U_i^b) is a neighborhood of a_i (respectively, b_i) in $K_{1,i}$, and U_i is a neighborhood of the set $\operatorname{supp} x_i \cap \pi_1^{-1}(\{c_j \mid j \neq l\})$. We can suppose that U_i^a, U_i^b, U_i are disjoint. Then the neighborhood $\{\langle U_2^a \times U_2^b \times U_2 \rangle \times \langle U_3^a \times U_3^b \times U_3 \rangle\} \cap B_{x_1}$ of the couple (x_2, x_3) in B_{x_1} is formed by points of B_{x_1}, satisfying property 1) at $m = l$, i.e., equals $M_{x_1}^l$. Thus $M_{x_1}^l$ is open in B_{x_1}. Now prove the connectedness of $M_{x_1}^l$. Denote by K^m the set $\pi_1^{-1}(c_m)$. Properties 1) and 2) mean that

$$M_{x_1}^l = A_2(K^l) \times A_2(K_l) \times \prod_{m \neq l} [(\exp K^m \times K^m) \cup_{K^m \times K^m} (\exp K^m \times K^m)]$$

(here $\cup_{K^m \times K^m}$ is the operation of gluing along $K^m \times K^m$).

By Lemma 5.1.21, all factors of this product are connected. Therefore, so is $M^l_{x_1}$. □

Proof of Theorem 5.1.8. Using Lemma 5.1.14, obtain that there exist an isomorphism $\mathcal{H}$ of the diagrams $\exp \pi^3(K^\omega)$ and $\exp \pi^3(L^\omega)$ and a point $x_1 \in \exp K_1$ with $\deg x_1 = \deg x$ and $\deg h_1(x_1) = \deg h(x)$. Besides, it follows from Lemma 5.1.19 that $B_{x_1}(K)$ and $B_{h(x_1)}(L)$ are homeomorphic. Then, by Lemma 5.1.20 we have $\deg x_1 = \deg h_1(x_1)$, i.e., $\deg x = \deg h(x)$. □

For every $x_1 \in \exp K_1$ put

$$C_{x_1}(K) = \{x_2 \in (\exp \pi_1)^{-1}(x_1) \mid \text{for all } x_3 \in (\exp \pi_1)^{-1}(x_1) \text{ one has } |\chi(x_2, x_3)| = 1\}.$$

Lemma 5.1.22. *Under the isomorphism $\mathcal{H}$ of the diagrams $\exp \pi^3(K^\omega)$ and $\exp \pi^3(L^\omega)$ we have $h_{12}(C_{x_1}(K)) = C_{h_1(x_1)}(L)$.*

Lemma 5.1.23. *The following*

$$C_{x_1} = \{x_2 \mid \pi_1|\operatorname{supp} x_2 \text{ is a one-to-one map onto } \operatorname{supp} x_1\}$$

holds.

Lemma 5.1.24. *Let y be the limit of a sequence $(y_i)_{i=1}^\infty$ from $\exp X$, x the limit of a sequence $(x_i)_{i=1}^\infty$, and $x_i \in \operatorname{supp} y_i$. Then $x \in \operatorname{supp} y$.*

Lemma 5.1.25. *Let K^ω be a stretchable compact metrizable space. Then C_{x_1} is close in $\exp K_{12}$ iff the degree of x_1 is finite.*

Proof. Suppose first that $\operatorname{supp} x_1 = \{c_1, \dots, c_2\}$. Then by Lemma 5.1.23 $C_{x_1} = \prod_{a \le m \le n} K^m$. Hence, C_{x_1} is compact and closed in $\exp K_{12}$.

Now let $\deg x_1 = \infty$. Let c_0 be a some limit point of $\operatorname{supp} x_1$, and $\{c_i\}_{i=1}^\infty$ converging to c_0 sequence in $\operatorname{supp} x_1$. Define a sequence $\{y_i\} \subset C_{x_1}$ in the following manner. Since K^ω is stretchable, there exists a sequence of functions $f_i \colon K_1 \to K_2$ with $\varrho(f_i(c_0), f_i(c_i)) > \varepsilon$, where K_2 is the second factor of K_{12} and ε is the constant from Definition 5.1.9. Then y_i is a point of $\exp K_{12}$ which support is formed by the intersection of the graph of f_i and $\pi_1^{-1}(\operatorname{supp} x_1)$. By Lemma 5.1.23 we have $y_i \in C_{x_1}$. Denote by z_i ($\overline{z}_i$) a couple $(c_0, f_i(c_0))$ (respectively,

$(c_i, f_i(c_i))$. By construction, $z_i, \overline{z}_i \in \operatorname{supp} y_i$. Since K_{12} and $\exp K_{12}$ are compact, there exists a sequence of indices i_k such that $\{y_{i_k}\}$ is converging in $\exp K_{12}$, and sequences $\{z_{i_k}\}$, $\{\overline{z}_{i_k}\}$ are convergent in K_{12}. Since $\lim c_{i_k} = c_0$ and $\varrho(f_{i_k}(c_{i_k}), f_{i_k}(c_0)) > \varepsilon$, the limits of $\{z_{i_k}\}$, $\{\overline{z}_{i_k}\}$ are of the forms (c_0, a_1) and (c_0, a_2), respectively. Besides, $\varrho(a_1, a_2) \geq \varepsilon$. Applying Lemma 5.1.24, obtain $(c_0, a_1), (c_0, a_2) \in \operatorname{supp} y$, where y is the limit of $\{y_{i_k}\}$. By Lemma 5.1.23, $y \notin C_{x_1}$. Hence, C_{x_1} is not closed in $\exp K_{12}$. □

Proof of Theorem 5.1.10. Let X be a point of finite degree in $\exp K^\tau$. By Lemma 5.1.14, we obtain that there exist an isomorphism $\mathcal{H}$ of the diagrams $\pi^3(K^\omega)$ and $\pi^3(L^\omega)$ and a point $x_1 \in \exp K_1$ with $\deg x_1 = \deg x$ and $\deg h_1(x_1) = \deg h(x)$. By Lemmas 5.1.22 and 5.1.25, $C_{h_1(x_1)}(L)$ is closed in $\exp L_{12}$. Then the degree of $h_1(x_1)$ is finite. Since $\deg h_1(x_1) = \deg h(x)$, we have $\deg h(x) < \infty$. □

Examples.

1. There exist a homeomorphism $h\colon \exp^2 I^\tau \to \exp^2 I^\tau$ and a point $A \in \exp^2 I^\tau$ such that $\deg(A) = 3$ and $\deg(h(A)) = 4$.

Indeed, consider the set $K = \{(x, y) \in I^2 \mid x \leq y\}$ and let $\varphi\colon K \to K$ be a homeomorphism such that $\varphi|\Delta = 1_\Delta$ (here $\Delta = \{(x, x) \mid x \in I\}$ is the diagonal) and $\varphi(0, 1/2) = (0, t_1)$, $\varphi(1/2, 1) = (t_2, 1)$, where $t_1 < 1/2 < t_2$. Denote by $\alpha_{\varphi,(x,y)}$ a nondecreasing affine map that maps the segment $[x, y]$ onto the segment $[x_1, y_1]$, where $(x_1, y_1) = \varphi(x, y)$. Besides, define a map $g\colon \exp(I \times I^\tau) \to K$ by the formula:

$$g(A) = (\min \exp \operatorname{pr}_1(A), \max \exp \operatorname{pr}_1(A)),\ A \in \exp(I \times I^\tau)$$

(by $\operatorname{pr}_1\colon I \times I^\tau \to I^\tau$ we denote the projection map.

Supposing τ infinite, we will consider the space $I \times I^\tau$ instead of I^τ. Let $a, b, c \in \exp(I \times I^\tau)$ be such that $\operatorname{pr}_1(a) = 0$, $\operatorname{pr}_1(b) = 1/2$, $\operatorname{pr}_1(c) = 1$. Define a map $h'\colon \exp(I \times I^\tau) \to \exp(I \times I^\tau)$ by the formula:

$$h'(A) = \exp(\alpha_{\varphi, g(A)} \times 1_{I^\tau})(A),\ A \in \exp(I \times I^\tau). \tag{5.2}$$

To see that h' is a homeomorphism note that the inverse map to h' is given by the formula

$$h'^{-1}(A) = \exp(\alpha_{\varphi^{-1}, g(A)} \times 1_{I^\tau}(A),\ A \in \exp(I \times I^\tau).$$

Then the map $h = \exp h'$ is an autohomeomorphism of the space $\exp^2(I \times I^\tau)$ which sends the point $\{\{a, b\}, \{b, c\}$ of degree 3 to a point of degree 4.

2. There exists a homeomorphism $h'\colon \exp I^\tau \to \exp I^\tau$ such that $h'(I^\tau) \neq I^\tau$.

In the notation of the preceding example, let $\varphi\colon K \to K$ be a homeomorphism such that $\varphi|\Delta = 1_\Delta$ and $\varphi(0, 1) \neq (0, 1)$. Define h' by formula (5.2). Then $\exp \operatorname{pr}_1(h'(I \times I^\tau)) = [x_1, y_1]$, where $(x_1, y_1) \neq (0, 1)$. Thus, $h'(I \times I^\tau) \neq I \times I^\tau$.

5.1.4. Spaces of probability measures of uncountable powers of compact metric spaces

Now we pass to the case of probability measure functor. Similarly as above, one can reduce the investigations of the spaces $P(K^\tau)$ to the properties of the diagrams $P(\pi^3 K)$.

Let $\mu_2, \mu_3 \in P(K \times K)$ be such that $P\pi_1(\mu_2) = P\pi_1(\mu_3)$. Then we put

$$\chi_K(\mu_2, \mu_3) = \{\mu \in P(K \times K \times K) \mid P\pi_{1i}(\mu) = \mu_i,\ i = 2, 3\}.$$

Lemma 5.1.26. *Suppose that $(\mu_2, \mu_3) \in P(K \times K) \times P(K \times K)$ is such that $P\pi_1(\mu_2) = P\pi_1(\mu_3)$ is of degree 1 and $\deg \mu_2 = n$, $\deg \mu_3 = m$. Then $\dim(\chi_K(\mu_2, \mu_3)) = (n-1)(m-1)$.*

Proof. Obvious. □

Lemma 5.1.27. *Suppose that $(\mu_2, \mu_3) \in P(K \times K) \times P(K \times K)$ is such that $P\pi_1(\mu_2) = P\pi_1(\mu_3) = \mu_1$, where $\deg \mu_1 = n$, $\mu_1 = \sum_{i=1}^n \alpha_i \delta_{x_i}$. Then*

$$\dim(\chi_K(\mu_2, \mu_3)) = \sum_{i=1}^n \dim(\chi_K(\frac{1}{\alpha_i}\mu_2|\pi_1^{-1}(x_i), \frac{1}{\alpha_i}\mu_3|\pi_1^{-1}(x_i))).$$

Proof. This is a consequence of Lemma 5.1.26. □

Lemma 5.1.28. *The space $\{\mu \in PX \mid \deg \mu = n\}$ is connected for every connected space X.*

Proof. We prove this only for the class of polyhedra. Let $\mu_1, \mu_2 \in PX$, $\deg(\mu_1) = \deg(\mu_2) = n$. There is a set $J \subset X$ which is homeomorphic to the segment and contains the set $\operatorname{supp}(\mu_1) \cup \operatorname{supp}(\mu_2)$. Thus, $\mu_1, \mu_2 \in PJ \subset PX$ and this reduces the problem to the case of the segment. We identify J with the segment $[0,1]$ and let $\operatorname{supp}(\mu_1) = \{x_1, \dots, x_n\}$, $\operatorname{supp}(\mu_2) = \{y_1, \dots, y_n\}$, where

$$x_1 < x_2 < \dots < x_n,\ y_1 < y_2 < \dots < y_n.$$

If $\mu_1 = \sum_{i=1}^n \alpha_i \delta_{x_i}$, $\mu_2 = \sum_{i=1}^n \beta_i \delta_{y_i}$, then define the path $f\colon [0,1] \to P[0,1]$ by the formula

$$f(t) = \sum_{i=1}^n (t\alpha_i + (1-t)\beta_i)\delta_{tx_i + (1-t)y_i}.$$

Clearly, f connects μ_1 and μ_2 in $P_n[0,1] \setminus P_{n-1}[0,1]$. □

Lemma 5.1.29. *Let*

$$\begin{aligned} B_{\mu_1}(K) =&\{(\mu_2, \mu_3) \in P(K \times K) \times P(K \times K) \mid P\pi_1(\mu_2) \\ =& P\pi_1(\mu_3) = \mu_1 \text{ and } \dim(\chi_K(\mu_2, \mu_3)) = 1\}. \end{aligned}$$

Then connectedness of $B_{\mu_1}(K)$ is equivalent to connectedness of K and the condition $\deg \mu_1 = 1$. If K is connected and $\deg \mu_1 = n$, then $B_{\mu_1}(K)$ consists of exactly n connected components.

Theorem 5.1.30. *Let K be a metrizable continuum; $\tau > \omega_1$. For every ahtohomeomorphism h of the space $P(K^\tau)$ we have $h(P_n(K^\tau)) = P_n(K^\tau$.*

Exercise

1. Reproduce all the details of the proof of Theorem 5.1.30.

5.2. Homeomorphisms of functor-powers

Theorem 5.2.1. *Let τ be an uncountable cardinal and F a normal functor of finite degree. If the spaces 2^τ and $F(2^\tau)$ are homeomorphic, then the functor F is isomorphic to the functor SP^n_G for some subgroup G of the symmetric group S_n.*

Proof. Using the Shchepin spectral theorem, we can deduce that the maps $\mathrm{pr}_1 \colon 2^\omega \times 2^\omega \to 2^\omega$ and $F\mathrm{pr}_1 \colon F(2^\omega \times 2^\omega) \to F(2^\omega)$ are homeomorphic. Thus, the map $F\mathrm{pr}_1 \colon F(2^\omega \times 2^\omega) \to F(2^\omega)$ is open and, by Proposition 2.10.10, the functor F is open. It remains to apply Theorem 2.10.21. □

Theorem 5.2.2. *Suppose that F is either a finite normal functor or a normal functor of finite degree such that $F(I_1^\omega)$ is an absolute retract. Then F is a power functor.*

Proof. E. Shchepin [1978] proved that any absolute retract can be represented as the limit space of a regular well-ordered inverse system with soft bonding maps. Applying the Shchepin spectral theorem, we see that the map $F\,\mathrm{pr}_1 \colon F(I^\omega \times I^\omega) \to F(I^\omega)$ is soft. Now the result follows from Theorems 4.2.11 and 4.2.12. □

Theorem 5.2.3. *Let $n, k \in \{1,3,4,5,\dots\}$, $n \neq k$. Then the spaces $\lambda_k 2^{\omega_1}$ and $\lambda_n 2^{\omega_1}$ are not homeomorphic.*

Proof. We suppose that $n < k$. Represent 2^{ω_1} as the limit space of the inverse system$\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$, where $X_0 = 2^\omega$, $X_{\alpha+1} = X_\alpha \times 2^\omega$ and $p_{\alpha+1,\alpha} \colon X_{\alpha+1} = X_\alpha \times 2^\omega \to X_\alpha$ are the projections. If the spaces $\lambda_k 2^{\omega_1}$ and $\lambda_n 2^{\omega_1}$ are homeomorphic, then, by the Shchepin spectral theorem, their homeomorphism induces an isomorphism of closed cofinal subsystems of the inverse systems $\lambda_n\mathcal{S}$ and $\lambda_k\mathcal{S}$. Since every cofinal closed subsystem of $\mathcal{S}$ is evidently isomorphic to φ, we may assume that $\lambda_n\mathcal{S}$ and $\lambda_k\mathcal{S}$ are isomorphic. Let $(f_\alpha \colon \lambda_n X_\alpha \to \lambda_k X_\alpha)_{\alpha\in\omega_1}$ be an isomorphism.

Show that then $f_\alpha(\lambda_{n-1}X_\alpha) = \lambda_{k-1}X_\alpha$, $\alpha < \omega$. Indeed, for every m the following holds $\mathcal{M} \in \lambda_m X_\alpha \setminus \lambda_{m-1}X_\alpha$ if and only if for every $\mathcal{N} \in (\lambda_m p_{\alpha+1,\alpha})^{-1}(\mathcal{M})$ and every neighborhood U of $\mathcal{N}$ the element $\mathcal{M}$ is an interior point of the set $\lambda_m p_{\alpha+1,\alpha}(U)$.

Thus, the map $f_\alpha|\lambda_{n-1}X_\alpha$ is a homeomorphism of $\lambda_{n-1}X_\alpha$ and $\lambda_{k-1}X_\alpha$. Repeating these arguments $(n-2)$ times, we see that

$$\lambda_{k-(n-2)}2^{\omega_1} \cong \lambda_2 2^{\omega_1} = \lambda_1 2^{\omega_1} \cong 2^{\omega_1}.$$

Shchepin spectral theorem, implies that the functor $\lambda_{k-(n-2)}$ is open. This contradicts to Proposition 2.10.18. □

Exercise

1. Find the counterparts of Theorem 5.2.3 for another functors of finite degree.

Problems

1. (*E. Shchepin*) We say that a map $f \colon X \to Y$ satisfies the *homeomorphism lifting property* (HLP) if for every homeomorphism $h \colon Y \to Y$ there exists a homeomorphism $g \colon X \to X$ such that $h \circ f = f \circ g$.
   Let F be a normal functor in **Comp** and $\mathrm{pr}_1 \colon I^\tau \times I^\tau$ the projection, where $\tau \geq \omega_2$.
Does the map $F\mathrm{pr}_1$ satisfy HLP?
2. Suppose that $h \colon \exp I^\tau \to \exp I^\tau$ is a homeomorphism, $\tau > \omega_1$. Is $w(h(A)) = w(A)$ for every $A \in \exp I^\tau$?

5.3. Geometry of multiplications of weakly normal monads

In this section we investigate geometry of multiplications of the inclusion hyperspace monad $\mathbb{G} = (G, \eta, \widehat{\mu})$, the full linked system monad $\mathbb{N} =$

(N, η, μ), and the superextension monad $\mathbb{L} = (\lambda, \eta, \mu)$.

A compact Hausdorff space is called *openly generated* if it can be represented as the limit space of an open inverse σ-system.

The class of openly generated spaces is the minimal class of compact Hausdorff spaces that contains all compact metrizable spaces and is closed under the operation of passing to the limit of well-ordered open inverse systems.

5.3.1. Multiplication of the superextension monad

Here we consider the case of monad $\mathbb{L} = (\lambda, \eta, \mu)$.

Theorem 5.3.1. *Let X be a continuum. Then the following conditions are equivalent:*
1) X is openly generated;
2) the map μX is soft.

Theorem 5.3.2. *Let X be a compact Hausdorff space. Then the following conditions are equivalent:*
1) X is openly generated;
2) the map μX is 0-soft.

For proving these theorems we need some notations. Recall that for $A \subset X$ we denote by A^+ the set $\{\mathcal{C} \in \lambda X \mid C \subset A \text{ for some } C \in \mathcal{C}\}$. For every points $x_1, x_2, x_3, y \in X$ and sets $A_1, A_2, A_3, B \subset X$ put

$$\begin{aligned}
t(x_1, x_2, x_3) &= \{\mathcal{C} \in \lambda X \mid |\mathcal{C} \cap \{x_1, x_2, x_3\}| \geq 2\}, \\
t(A_1, A_2, A_3) &= \{t(a_1, a_2, a_3) \mid x_i \in a_i,\ i = 1, 2, 3\}, \\
\tilde{t}(A_1, A_2, A_3) &= (A_1 \cup A_2)^+ \cup (A_1 \cup A_3)^+ \cup (A_2 \cup A_3)^+, \\
s(y; x_1, x_2, x_3) &= \{\mathcal{C} \in \lambda X \mid \mathcal{C} \supset \{x_1, x_2, x_3\} \text{ and } \mathcal{C} \supset \{y, x_i\}, \\
&\qquad i = 1, 2, 3\}, \\
s(B; A_1, A_2, A_3) &= \{s(b; a_1, a_2, a_3) \mid b \in B,\ a_i \in A_i,\ i = 1, 2, 3\}, \\
\tilde{s}(B; A_1, A_2, A_3) &= (B \cup A_1)^+ \cup (B \cup A_2)^+ \cup (B \cup A_3)^+ \\
&\qquad \cup (A_1 \cup A_2 \cup A_3)^+, \\
u(x_1, x_2, x_3; y) &= t\big(s(x_1; x_2, x_3, y), s(x_2; x_1, x_3, y), s(x_3; x_1, x_2, y)\big), \\
u(A_1, A_2, A_3; B) &= \{u(a_1, a_2, a_3, b) \mid b \in B,\ a_i \in A_i,\ i = 1, 2, 3\}, \\
\tilde{u}(A_1, A_2, A_3; B) &= \tilde{t}\big(\tilde{s}(A_1; A_2, A_3, B), \tilde{s}(A_2; A_1, A_3, B), \\
&\qquad \tilde{s}(A_3; A_1, A_2, B)\big).
\end{aligned}$$

Remark some properties of $\tilde{u}$ and u.

Lemma 5.3.3. *1) $\mu X\big(u(x_1, x_2, x_3; y)\big) = t(x_1, x_2, x_3)$;*

2) for every disjoint subsets A_1, A_2, A_3, B of X and every element $\mathfrak{M} \in \tilde{u}(A_1, A_2, A_3; B)$ we have $\mathrm{supp}_{\lambda^2}(\mathfrak{M}) \cap B \neq \varnothing$;

3) let a_1, a_2, a_3, b be distinct points of X and $H \subset \lambda^2 X$ an open set, containing $u(a_1, a_2, a_3; b)$; then there exist neighborhoods Oa_1, Oa_2, Oa_3, Ob of points a_1, a_2, a_3, b respectively such that

$$u(Oa_1, Oa_2, Oa_3, Ob) \subset H;$$

4) for every subsets A_1, A_2, A_3, B of X and map $f\colon X \to Y$ we have $\lambda^2\big(\tilde{u}(A_1, A_2, A_3; B)\big) \subset \tilde{u}\big(f(A_1), f(A_2), f(A_3); f(B)\big)$.

Proof. For 1) we have

$$\{x_1, x_2\} \in s(x_1; x_2, x_3, y) \cap s(x_2; x_1, x_3, y) \subset \mu X\big(u(x_1, x_2, x_3; y)\big).$$

Similary, $\{x_1, x_3\}$ and $\{x_2, x_3\}$ also belong to $\mu X\big(u(x_1, x_2, x_3; y)\big)$.

To the contrary of 2), $\mathrm{supp}(\mathfrak{M}) \cap B = \varnothing$. Then $\mathfrak{M} \in \lambda^2 C$ for some compact $C \subset X \setminus B$. Considering some $\mathcal{B} \in \mathfrak{M}$ with

$$\mathcal{B} \subset \tilde{s}(A_1; A_2, A_3, B) \cup \tilde{s}(A_2; A_1, A_3, B) \text{ and } \mathcal{B} \subset \lambda C,$$

obtain contradiction with

$$\big(\tilde{s}(A_1; A_2, A_3, B) \cup \tilde{s}(A_2; A_1, A_3, B)\big) \cap \lambda C = \varnothing.$$

Since the maps $t\colon X^3 \to \lambda X$ and $s\colon X^4 \to \lambda X$ is continuous for all X, we have continuity of u and hence, statement 3) holds.

And finally, 4) is obvious. $\square$

Proof of Theorem 5.3.1. 1)$\Rightarrow$2) Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, hook] & \lambda^2 X \arrow[d, "\mu X"] \\ Z \arrow[r, "\varphi"] & \lambda X, \end{tikzcd}$$

where Z is a paracompact Hausdorff space and A a closed subset of Z. Since λX is an AR for openly generated X (see A. Ivanov [1981]), there

exists an extension $\Phi\colon Z \to \lambda^2 X$ of ψ. Recall that the nearest point map $\xi X\colon \lambda X \times N_2 X \to \lambda X$,

$$\xi X(\mathcal{M}, \mathcal{N}) = \mathcal{N} \cup \{M \in \mathcal{M} \mid M \cap N \neq \varnothing \text{ for all } N \in \mathcal{N}\}$$

is continuous and satisfies the following: $\xi X(\mathcal{M}, \mathcal{N}) \supset \mathcal{N}$. Now, for a map $\Phi_1\colon Z \to \lambda^2 X$,

$$\Phi_1(z) = \xi X\left(\Phi(z), \mu X^{-1}(\varphi(z))\right)$$

we have $\Phi_1|A = \psi$ and $\mu X \circ \Phi_1 = \varphi$.

2)$\Rightarrow$1) Let X be a continuum with $w(X) = \tau > \omega$. Represent X as the limit of an inverse σ-system $\mathcal{S} = \{X_\alpha, p_\beta^\alpha; \mathcal{P}_\omega(\tau)\}$. Let $\mathcal{M}, \mathcal{N} \in \lambda X$, $\deg(\mathcal{M}) = \deg(\mathcal{N}) = 3$, and $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{N}) = \varnothing$.

Put $\mathcal{M}_\alpha = \lambda p_\alpha(\mathcal{M})$, $\mathcal{N}_\alpha = \lambda p_\alpha(\mathcal{N})$, $\alpha \in \mathcal{P}_\omega(\tau)$. Without restriction of generality we can suppose that $\deg(\mathcal{M}_\alpha) = \deg(\mathcal{N}_\alpha) = 3$ and $\operatorname{supp}(\mathcal{M}_\alpha) \cap \operatorname{supp}(\mathcal{N}_\alpha) = \varnothing$. Consider the following compacta:

$$\begin{aligned} A &= (\mu X)^{-1}(\mathcal{M}), & B &= (\mu X)^{-1}(\mathcal{N}), \\ A_\alpha &= (\mu X_\alpha)^{-1}(\mathcal{M}_\alpha), & B_\alpha &= (\mu X_\alpha)^{-1}(\mathcal{N}_\alpha), \quad \alpha \in \mathcal{P}_\omega(\tau). \end{aligned}$$

By naturality of μ we see that $\lambda^2 p_\beta^\alpha(A_\alpha) = A_\beta$, $\lambda^2 p_\beta^\alpha(B_\alpha) = B_\beta$, and

$$A = \varprojlim\{A_\alpha, \lambda^2 p_\beta^\alpha|A_\alpha; \mathcal{P}_\omega(\tau)\}, \quad B = \varprojlim\{B_\alpha, \lambda^2 p_\beta^\alpha|B_\alpha; \mathcal{P}_\omega(\tau)\}.$$

Since μX is soft, we have $A, B \in \mathrm{AR}$. Therefore, by the Schepin spectral theorem, we can suppose that $\lambda^2 p_\beta^\alpha|A_\alpha$ and $\lambda^2 p_\beta^\alpha|B_\alpha$ are open. Show that this fact implies openness of p_β^α.

Otherwise, there exists an open set $U \subset X_\alpha$ such that $p_\beta^\alpha(U)$ is not open on X_β. Moreover, we can suppose that $p_\beta^\alpha(\overline{U}) \cap \operatorname{supp}(\mathcal{M}_\beta) = \varnothing$. Then $\overline{U} \cap \operatorname{supp}(\mathcal{M}_\alpha) = \varnothing$.

Let $\operatorname{supp}(\mathcal{M}_\gamma) = \{x_\gamma, y_\gamma, z_\gamma\}$ for all $\gamma \in \mathcal{P}_\omega(\tau)$. Choose neighborhoods $Ox_\alpha, Ox_\alpha, Ox_\alpha$ of points $x_\alpha, y_\alpha, z_\alpha \in X_\alpha$, respectively, such that a system $p_\beta^\alpha(\overline{Ox_\alpha})$, $p_\beta^\alpha(\overline{Oy_\alpha})$, $p_\beta^\alpha(\overline{Oz_\alpha})$, $p_\beta^\alpha(\overline{U})$ is disjoint.

Put $V = A_\alpha \cap \tilde{u}(Ox_\alpha, Oy_\alpha, Oz_\alpha; U)$. Then V is an open set in A_α. Therefore, $\lambda^2 p_\beta^\alpha(V)$ is open in A_β.

Let $w \in U$ be a point such that the point $p_\beta^\alpha(w)$ is not internal for $p_\beta^\alpha(U)$. By statement 1) of Lemma 5.3.3 we have

$$u(x_\beta, y_\beta, z_\beta; p_\beta^\alpha(w)) \in \lambda^2 p_\beta^\alpha(V).$$

By statement 3) for this lemma there exist disjoint neighborhoods Ox_β, Oy_β, Oz_β, Ow of points $x_\beta, y_\beta, z_\beta, w$, respectively, such that

$$u(Ox_\beta, Oy_\beta, Oz_\beta; Ow) \subset \lambda^2 p^\alpha_\beta(V).$$

Let $v \in Ow \setminus p^\alpha_\beta(U)$. Put $\mathfrak{M} = u(x_\beta, y_\beta, z_\beta; v)$. Let $\mathfrak{N} \in V$ be a point such that $\lambda^2 p^\alpha_\beta(\mathfrak{N}) = \mathfrak{M}$. By statement 2) of Lemma 5.3.3 obtain $\operatorname{supp}(\mathfrak{N}) \cap U \neq \varnothing$ and $\operatorname{supp} \lambda^2 p^\alpha_\beta(\mathfrak{N}) \cap p^\alpha_\beta(U) \neq \varnothing$. But

$$\begin{aligned}\operatorname{supp} \lambda^2 p^\alpha_\beta(\mathfrak{N}) &= \operatorname{supp}(\mathfrak{M}) \\ &= \{x_\beta, y_\beta, z_\beta, v\} \text{ and } \{x_\beta, y_\beta, z_\beta, v\} \cap p^\alpha_\beta(U) = \varnothing.\end{aligned}$$

Contradiction. □

A *trivial I^τ-fibration* is a map of the form $\mathrm{pr}_1 : X \times I^\tau \to X$. A map $f : X \to Y$ is a *locally trivial I^τ-fibration* if for every every point $y \in Y$ there is a neighborhood U of y such that the map $f|f^{-1}(U) : f^{-1}(U) \to U$ is a trivial I^τ-fibration.

The following is a characterization theorem for I^ω-fibrations (Q-fibrations).

A map $f : X \to Y$ of compact metric spaces satisfies the *disjoint approximation condition* if for every $\varepsilon > 0$ there exist ε-close to id_X maps $g_1, g_2 : X \to X$ such that $g_1(X) \cap g_2(X) = \varnothing$, $f \circ g_i = f$, $i = 1, 2$.

Theorem 5.3.4. (Toruńczyk-West theorem.) *A soft map $f : X \to Y$ of metrizable AR-compacta is homeomorphic to a trivial Q-fibration if and only if it satisfies the disjoint approximation condition.*

Let $\lambda_\nabla X = \lambda X \setminus \eta X(X)$, $\lambda^2_\nabla X = \lambda^2 X \setminus \eta\lambda X(\eta X(X))$. Then we have $\mu X(\lambda^2_\nabla X) = \lambda_\nabla X$. Put $\mu_\nabla X = \mu X|\lambda^2_\nabla X : \lambda^2_\nabla X \to \lambda_\nabla X$.

Proposition 5.3.5. *For every map $f : X \to Y$ in* **Comp** *the diagram*

$$\begin{array}{ccc} \lambda^2 X & \xrightarrow{\mu X} & \lambda X \\ {\scriptstyle \lambda^2 f}\downarrow & & \downarrow{\scriptstyle \lambda f} \\ \lambda^2 Y & \xrightarrow{\mu Y} & \lambda Y \end{array}$$

is bicommutative.

Proof. It is sufficient to prove that for every $\mathfrak{M} \in \lambda^2 Y$, $\mathcal{M} \in \lambda X$ such that $\mu Y(\mathfrak{M}) = \lambda f(\mathcal{M})$ the set

$$(\lambda^2 f)^{-1}(\mathfrak{M}) \cap (\mu X)^{-1}(\mathcal{M}) = (\lambda^2 f)^{-1}(\mathfrak{M}) \cap (\cap\{M^{++} \mid M \in \mathcal{M}\})$$

is nonempty. Suppose the contrary, then there exist $M \in \mathcal{M}$ and $\mathcal{A} \in \mathfrak{M}$ such that $(\lambda f)^{-1}(\mathcal{A}) \cap M^+ = \varnothing$. This implies that for every $\mathcal{P} \in \mathcal{A}$ there exists $A \in \mathcal{P}$ such that $f^{-1}(A) \cap M = \varnothing$ and hence $A \cap f(M) = \varnothing$. This gives $\mathcal{A} \cap (f(M))^+ = \varnothing$. Since $f(M) \in \mathcal{N}$, we see that $\mathfrak{M} \notin \cap\{N^{++} \mid N \in \mathcal{N}\}$ and this gives a contradiction. □

A compact Hausdorff space is called *character homogeneous* if the characters of all its points are equal (recall that the *character* at a point is the minimal cardinality of base at this point).

Theorem 5.3.6. *Let X be a continuum of weight τ. The following conditions are equivalent:*

1) X is character homogeneous openly generated space;

2) the map $\mu_\nabla X$ is a locally trivial I^τ-fibration.

Proof. 1) $\Longrightarrow$ 2). We first consider the case of metrizable X. In this case, λX is homeomorphic to Q (J. van Mill [1980], A. Ivanov [1980]). It is sufficient to show that the map $\mu X|\mu X^{-1}(A)$ is a Q-fibration for every subset A of $\lambda_\nabla X$ homeomorphic to Q.

Fix $\varepsilon > 0$. There exist two disjoint sets $C_1, C_2 \subset X$ such that the sets $(\lambda C_i) \setminus \eta X(X)$ is an $\varepsilon/2$-net in λX. For every closed subset M in X let

$$S_i(M) = \{\xi(\mathcal{N}, M^+) \mid \mathcal{N} \in (\lambda C_i) \setminus \eta X(X)\},\ i = 1, 2,$$

and for every $\mathcal{M} \in \lambda X$ let

$$D_i(\mathcal{M}) = \cup\{S_i(M) \mid M \in \mathcal{M}\},\ i = 1, 2.$$

Evidently, $D_i \colon A \to \exp \lambda X$ are continuous maps.

We will need the following facts:

1). If $\mathcal{M} \in (\lambda X) \setminus \eta X(X)$, then $D_1(\mathcal{M}) \cap D_1(\mathcal{M}) = \varnothing$. Indeed, otherwise there exist $M_1, M_2 \in \mathcal{M}$ such that $S_1(M) \cap S_2(M) \neq \varnothing$ and therefore there exist $\mathcal{N}_i \in (\lambda C_i) \setminus \eta X(X)$, $i = 1, 2$, for which $\xi(\mathcal{N}_1, M_1^+) = (\mathcal{N}_2, M_2^+)$. Choose $x_i \in M_i$, $i = 1, 2$, so that $x_1 \neq x_2$. Then there exist $N_i \in \mathcal{N}_i$, $i = 1, 2$, such that $x_1 \notin N_2$, $x_2 \notin N_1$, and consequently $(N_1 \cup \{x_1\}) \cap (N_2 \cup \{x_2\}) = \varnothing$. However, $N_i \cup \{x_i\} \in \xi(\mathcal{N}_i, M_i^+)$, $i = 1, 2$, and we obtain a contradiction.

2). For every $M \in \mathcal{M}$ we have $D_i(\mathcal{M}) \cap M^+ \neq \varnothing$. Indeed, $\varnothing \neq S_i(M) \subset D_i(\mathcal{M}) \cap M^+$.

Define the maps $q_i \colon \lambda X \to \exp \lambda X$ by $q_i(\mathcal{M}) = \cap\{M^{++} \mid M \in \mathcal{M}\} \cap D_i(\mathcal{M}^+)$, and let $s_i \colon \mu X^{-1}(A) \to \mu X^{-1}(A)$ be a map defined by $s_i(\mathfrak{M}) = \xi(\mathfrak{M}, q_i \mu X^{-1}(\mathfrak{M}))$. Then

$$s_i(\mathfrak{M}) = \cap\{M^{++} \mid M \in \mu X(\mathfrak{M})\} = \mu X^{-1}(\mathfrak{M})$$

and therefore we have

3). $\mu X = \mu X \circ s_i$, $i = 1, 2$.

This fact and $D_1(\mu X(\mathfrak{M}) \cap D_2(\mu X(\mathfrak{M}) = \varnothing$ imply $s_1(\mu X^{-1}(A)) \cap s_2(\mu X^{-1}(A)) = \varnothing$.

4). $d(s_i, 1_{\mu X^{-1}(A)}) \leq \frac{\varepsilon}{2}$. Indeed, since $\lambda C_i \setminus \nu X(X)$is an $\frac{\varepsilon}{2}$-net in M^{++} and ξ is a nonexpanding map, $S_i(M^+)$ is also an $\frac{\varepsilon}{2}$-net in M^{++} for every $M \in \mu X(\mathfrak{M})$. Let $\mathcal{A} \in \mathfrak{M}$, then $\mathcal{A} \cap M^{++} \neq \varnothing$. As we have shown, $O_{\frac{\varepsilon}{2}}(\mathcal{A}) \cap S_i(M^+) \neq \varnothing$. Hence $O_{\frac{\varepsilon}{2}}(\mathcal{A}) \cap D_i(\mu X(\mathfrak{M})) \neq \varnothing$ and consequently $B_{\frac{\varepsilon}{2}}(\mathcal{A}) \in s_i(\mathfrak{M})$.

Summing up, we see that the map $\mu X|\mu X^{-1}(A)$ satisfies the disjoint approximation condition. Being soft, as the restriction of a soft map onto a full preimage, by The Toruńczyk-West theorem, $\mu X|\mu X^{-1}(A)$ is a trivial Q-fibration.

Now suppose that $\mathrm{w}(X) = \tau > \omega$ and $X = \varprojlim \mathcal{S}$ for a regular open inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \tau\}$. We may suppose that $X_0 = \{*\}$ and that the preimages of $p_{\alpha\beta}$ are not singletons for $\alpha > \beta$ (the latter is a consequence of the fact that X is character homogeneous). Let $\mathcal{M} \in \lambda_\nabla X$, then there exist closed sets $A_1, A_2, A_3 \subset X$ such that $A_1 \cap A_2 \cap A_3 = \varnothing$ and $\mathcal{M} \in \mathrm{Int}(A_1^+ \cap A_2^+ \cap A_3^+)$. Let $K = A_1^+ \cap A_2^+ \cap A_3^+$. It is sufficient to show that the map $\mu X|\mu X^{-1}(K)$ is a trivial I^τ-fibration.

For every $\alpha < \tau$ let

$$\begin{aligned} K_\alpha &= p_\alpha(A_1)^+ \cap p_\alpha(A_2)^+ \cap p_\alpha(A_3)^+, \\ L_\alpha &= \mu X_\alpha^{-1}(K_\alpha) \end{aligned}$$

and let $Z_\alpha = L_\alpha \times_{K_\alpha} K$ be the fibered product of the maps $\mu X_\alpha|L_\alpha$ and $\lambda p_\alpha|K \colon K \to K_\alpha$. For $\alpha > \beta$ denote by $t_{\alpha\beta} \colon Z_\alpha \to Z_\beta$ the map defined by the formula

$$t_{\alpha\beta}(\mathfrak{M}, \mathcal{M}) = (\lambda^2 p_{\alpha\beta}(\mathfrak{M}), \mathcal{M}), \ (\mathfrak{M}, \mathcal{M}) \in Z_\alpha.$$

There exists $\beta_0 < \tau$ such that $p_{\beta_0}(A_1) \cap p_{\beta_0}(A_2) \cap p_{\beta_0}(A_3) = \varnothing$. Show that for $\alpha > \beta \geq_{\beta_0}$ the map $t_{\alpha\beta}$ is soft and admits two disjoint sections.

Let $(\mathfrak{N}, \mathcal{M}) \in Z_\beta$. Then

$$\begin{aligned} t_{\alpha\beta}^{-1}(\mathfrak{N}, \mathcal{M}) =& \{(\mathfrak{N}, \mathcal{M}) \mid \lambda^2 p_{\alpha\beta}(\mathfrak{M}) = \mathfrak{N},\ \mu X_\alpha(\mathfrak{M}) = \lambda p_\alpha(\mathcal{M})\} \\ =& \{(\mathfrak{N}, \mathcal{M}) \mid \mathfrak{M} \in \cap\{(\lambda p_{\alpha\beta})^{-1}(A)^+ \mid A \in \mathfrak{N}\} \\ & \cap(\cap\{M^{++} \mid M \in \mathcal{M}\})\}. \end{aligned}$$

By Proposition 5.3.5, the sets $t_{\alpha\beta}^{-1}(\mathfrak{N}, \mathcal{M})$ are nonempty. As in the proof of Theorem 5.3.1 we conclude that the map $t_{\alpha\beta}$ is soft.

Define the maps $s_1, s_2 \colon Z_\beta \to Z_\alpha$ by the formulae

$$\begin{aligned} s_1(\mathfrak{N}, \mathcal{M}) =& (\xi(\eta\lambda X_\alpha \circ \lambda p_\alpha(\mathcal{M}), \lambda^2 p_{\alpha\beta}(\mathfrak{N})), \mathcal{M}), \\ s_2(\mathfrak{N}, \mathcal{M}) =& (\xi(\lambda\eta X_\alpha \circ \lambda p_\alpha(\mathcal{M}), \lambda^2 p_{\alpha\beta}(\mathfrak{N})), \mathcal{M}), \end{aligned}$$

where $(\mathfrak{N}, \mathcal{M}) \in Z_\beta$. Continuity of s_i is a consequence of openness of $p_{\alpha\beta}$.

We can find two disjoint sections $\zeta_1, \zeta_2 \colon \lambda X_\beta \to \lambda X_\alpha$ of the map $\lambda p_{\alpha\beta}$ such that $\mathcal{M}_\alpha = \lambda p_\alpha(\mathcal{M}) \in \zeta_1(\lambda X_\beta) \subset \lambda_\nabla X_\alpha$.

There exists a set $\mathcal{B} \in \lambda\zeta_1(\mathfrak{N})$ such that $\mathcal{M}_\alpha \in \mathcal{B}$. Then $\mathcal{B} \in \xi(\eta\lambda X_\alpha \circ \lambda p_\alpha(\mathcal{M}), \lambda^2 p_{\alpha\beta}^{-1}(\mathfrak{N}))$. Let $M \in \mathcal{M}_\alpha$, then

$$\mathcal{B}' = \zeta_2 \circ \lambda p_{\alpha\beta}(\mathcal{B}) \cup \eta X_\alpha(M) \in \xi(\eta\lambda X_\alpha \circ \lambda p_\alpha(\mathcal{M}), \lambda^2 p_{\alpha\beta}^{-1}(\mathfrak{N}))$$

and $\mathcal{B} \cap \mathcal{B}' = \varnothing$, whence $\mathcal{B} \notin \xi(\eta\lambda X_\alpha \circ \lambda p_\alpha(\mathcal{M}), \lambda^2 p_{\alpha\beta}^{-1}(\mathfrak{N}))$ and consequently, $s_1(\mathfrak{N}, \mathcal{M}) \neq s_2(\mathfrak{N}, \mathcal{M})$.

Applying arguments from A. Chigogidze [1986] and E. Shchepin [1978] we conclude that the map $\mu X | \mu X^{-1}(K)$ is a trivial I^τ-fibration.

$\square$

Let X, Y be compact Hausdorff spaces, A a closed subset of X. Define the *partial product* $P(X, Y; A)$. Denote by $\sim$ the following equivalence relation on $X \times Y$: $(a, y_1) \sim (a, y_2)$, $a \in A$, $y_1, y_2 \in Y$. Then $P(X, Y; A) = (X \times Y)/\sim$. Let $\langle x, y \rangle$ be an equivalence class of the point $(x, y) \in X \times Y$, $q \colon X \times Y \to P(X, Y; A)$ the quotient map. Denote by $\pi \colon P(X, Y; A) \to X$ the natural projection onto the first factor.

Given also a partial product $P(X'Y'; A')$ and maps $f \colon X \to X'$, $g \colon Y \to Y'$ with $f(A) \subset A'$ we define a map $P(f, g) \colon P(X, Y; A) \to P(X', Y'; A')$ by the formula $P(f, g)(\langle x, y \rangle) = \langle f(x), g(y) \rangle$. Obviously, the map $P(f, g)$ is continuous.

Theorem 5.3.7. *Let X be a character homogeneous openly generated continuum and $\tau = \omega(X)$. A map $\mu_\nabla X$ is homeomorphic to a trivial I^τ-fibration iff $\tau = \omega$.*

Proof. Necessity. If $\omega(X) = \omega$, then the space $\lambda_\nabla X$ is metrizable. Hence, it is paracompact and, therefore, by the Chapman theorem (see T. Chapman [1973]), the locally-trivial I^τ fiberation $\mu_\nabla X$ is trivial.

Sufficiency. Let $\omega(X) = \tau > \omega$. Suppose that $\mu_\nabla X$ is a trivial I^τ-fiberation. Then the map $\mu_\nabla X$ is homeomorphic to a projection $\pi = \pi X\colon P(\lambda X, I^\tau; \eta X(X)) \to \lambda X$. Now it is sufficient to show that the space $P(\lambda X, I^\tau; \eta X(X))$ is not an absolute retract. Indeed, then we shall obtain a contradiction with the following fact: $\lambda^2 X \in \mathrm{AR}$ for considering X (see A. Ivanov [1982]).

Let $\mathcal{S} = \{X_\alpha, p_{\alpha,\beta}; \mathcal{P}(\tau)\}$ be a sigma-system with $X = \varprojlim \mathcal{S}$ (here we denote by $\mathcal{P}_\omega(\tau)$ an index set of countable subsets of τ, ordering by the inclusion). Let $p_\alpha\colon X \to X_\alpha$ be an α-limit projection of this system. Let also $g_\alpha\colon I^\tau \to I^\alpha$ be a projection, $\alpha \in \mathcal{P}_\omega(\tau)$. Supposing that $P(\lambda X, I^\tau; \eta X(X)) \in \mathrm{AR}$, by the Shchepin spectral theorem obtain that there exists α for which a map

$$P(\lambda p_\alpha, g_\alpha)\colon P(\lambda X, I^\tau; \eta X(X)) \to P(\lambda X_\alpha, I^\alpha; \eta X_\alpha(X_\alpha))$$

is open.

But it is impossible. Indeed, let $V \neq \varnothing$ be an open subset of I^τ such that $g_\alpha(V) \neq I^\alpha$. Then $U = q(\lambda_\nabla X \times V)$ is an open subset of $P(\lambda X, I^\tau; \eta X(X))$. But its image $P(\lambda p_\alpha, g_\alpha)(U)$ is not open. This contradiction proves the theorem. □

One can prove a zero-dimensional counterpart of Theorem 5.3.7.

Theorem 5.3.8. *Let X be a homogeneous with respect to character openly generated continuum and $\tau = \omega(X)$. A map $\mu_\nabla X$ is homemorphic to a 2^τ-fibration iff $\tau = \omega$.*

5.3.2. Multiplication of the monads $\mathbb{G}$ and $\mathbb{N}$

Recall that the transversality map $\perp X\colon GX \to GX$ is defined by

$$\perp X(\mathcal{A}) = \{B \in \exp X \mid B \cap A \neq \varnothing \text{ for all } A \in \mathcal{A}\}, \quad \mathcal{A} \in GX.$$

To simplify the notation we shall write $\perp \mathcal{A}$ instead of $\perp X(\mathcal{A})$. Recall also that for $A \in \exp X$

$$\begin{aligned} A^+ &= \{\mathcal{A} \in GX \mid A \in \mathcal{A}\}, \\ A^- &= \{\mathcal{B} \in GX \mid B \cap A \neq \varnothing \text{ for all } B \in \mathcal{B}\}. \end{aligned}$$

Let $\mathcal{A}, \mathcal{B} \in \exp^2 X$. Define

$$\begin{aligned} \mathcal{H}(\mathcal{A}, C\mathcal{B}) &= \left(\bigcap\{A^+ \mid A \in \mathcal{A}\}\right) \cap \left(\bigcap\{B^- \mid B \in \mathcal{B}\}\right) = \\ &= \{\mathcal{C} \in GX \mid \mathcal{C} \supset \mathcal{A},\ \perp\mathcal{C} \supset \mathcal{B}\}. \end{aligned}$$

By analogy with the terminology in the theory of superextensions, sets of the form $\mathcal{H}(\mathcal{A}, \mathcal{B})$ will be called *convex*. Denote by KGX the subspace of $\exp GX$ consisting of nonempty convex subsets of GX. Obviously, $\mathcal{H}(\mathcal{A}, \mathcal{B}) \neq \varnothing$ if and only if $A \cap B \neq \varnothing$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let

$$Y = \left\{(\mathcal{A}, \mathcal{B}) \in \exp^2 X \times \exp^2 X \mid A \cap B \neq \varnothing \text{ for all } A \in \mathcal{A},\ B \in \mathcal{B}\right\}.$$

Lemma 5.3.9. *The map $\mathcal{H}\colon Y \to KGX$ is continuous.*

Proof. Let $\mathfrak{A}$ be an open subset of GX with $\mathcal{H}(\mathcal{A}, \mathcal{B}) \cap \mathfrak{A} \neq \varnothing$. If $\mathcal{C} \in \mathcal{H}(\mathcal{A}, \mathcal{B}) \cap \mathfrak{A}$, then there exist open sets $U_1, \ldots, U_k$ and $V_1, \ldots, V_l$ in X such that

$$\mathcal{C} \in U_1^+ \cap \cdots \cap U_k^+ \cap V_1^- \cap \cdots \cap V_l^- \subset \mathfrak{A}.$$

Then $\mathcal{A} \in \langle\langle X, V_1, \ldots, V_l\rangle\rangle$ and $\mathcal{B} \in \langle\langle X, V_1, \ldots, V_l\rangle\rangle$, and it is not hard to see that for any $(\mathcal{A}', \mathcal{B}') \in Y \cap (\langle\langle X, V_1, \ldots, V_l\rangle\rangle \times \langle\langle X, V_1, \ldots, V_l\rangle\rangle)$ we have $\mathcal{H}(\mathcal{A}', \mathcal{B}') \cap \mathfrak{A} \neq \varnothing$.

Assume now that $\mathcal{H}(\mathcal{A}, \mathcal{B}) \subset \mathfrak{A}$. Then there exist finite subfamilies $\{A_1, \ldots, A_m\} \in \mathcal{A}$ and $\{B_1, \ldots, B_n\} \in \mathcal{B}$ such that

$$A_1^+ \cap \cdots \cap A_m^+ \cap B_1^- \cap \cdots \cap B_n^- \subset \mathfrak{A}.$$

There exist open sets $\widetilde{U}_i \supset A_i$ and $\widetilde{V}_j \supset B_j$, $1 \leq i \leq m$, $1 \leq j \leq n$, such that

$$\widetilde{U}_1^+ \cap \cdots \cap \widetilde{U}_m^+ \cap \widetilde{V}_1^- \cap \cdots \cap \widetilde{V}_n^- \subset \mathfrak{A}.$$

Then $\mathcal{A} \in \langle \exp X, \langle \widetilde{U}_1\rangle, \ldots, \langle \widetilde{U}_m\rangle\rangle$ and $\mathcal{B} \in \langle \exp X, \langle \widetilde{V}_1\rangle, \ldots, \langle \widetilde{V}_n\rangle\rangle$, for any $(\mathcal{A}', \mathcal{B}') \in Y \cap \langle \exp X, \langle \widetilde{U}_1\rangle, \ldots, \langle \widetilde{U}_m\rangle\rangle \times \langle \exp X, \langle \widetilde{V}_1\rangle, \ldots, \langle \widetilde{V}_n\rangle\rangle$ we have $\mathcal{H}(\mathcal{A}', \mathcal{B}') \subset \mathfrak{A}$.

The continuity of $\mathcal{H}$ now follows from the definition of the base for the Vietoris topology on $\exp GX$. $\square$

We define the "nearest point" map $\xi_G: GX \times KGX \to GX$ by the formula

$$\xi_G(C, \mathcal{H}(\mathcal{A}, \mathcal{B})) = \mathcal{A} \cup (C \cap \perp \mathcal{B}).$$

It is not hard to show that xi_G is well defined, i.e.,

$$\xi_G(C, \mathcal{H}(\mathcal{A}, \mathcal{B})) = \xi_G(C, \mathcal{H}(\mathcal{A}', \mathcal{B}'))$$

whenever $\mathcal{H}(\mathcal{A}, \mathcal{B}) = \mathcal{H}(C\mathcal{A}', \mathcal{B}')$. Moreover,

$$\xi_G(C, \mathcal{H}(\mathcal{A}, \mathcal{B})) \in \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

Lemma 5.3.10. *The mapping ξ_G is continuous.* □

Lemma 5.3.11. *For a map $f: X \to Y$ and an $\mathcal{A} \in GX$,*

$$(Gf)^{-1}(\mathcal{A}) = \left(\bigcap\{(f^{-1}A)^+ \mid A \in C\mathcal{A}\}\right)\left(\bigcap\{(f^{-1}B)^- \mid B \in \perp\mathcal{A}\}\right).$$

The proof follows from Lemma 2.1.14 and the directly verifiable formulae $(Gf)^{-1}(A^+) = \left(f^{-1}(A)\right)^+$ and $(Gf)^{-1}(B^-) = \left(f^{-1}(B)\right)^-$.

Note that by Lemmas 3.2.7 and 5.3.11, the fibers of the maps $\mu_G X$ and Gf are convex.

Theorem 5.3.12. *The following conditions are equivalent for a continuum X:*

1) X is openly generated;
2) the map $\mu_G X: G^2X \to GX$ is soft;
3) the map $\mu_N X: N^2X \to NX$ is soft.

The proof follow the scheme of the proof of Theorem 5.3.1.

Theorem 5.3.13. *For a continuum X of weight τ the following conditions are equivalent:*

1) X is openly generated and character homogeneous;
2) the map $\mu_G X: G^2X \to GX$ is homeomorphic to the projection $\mathrm{pr}_1: I^\tau \times I^\tau \to I^\tau$;
3) the map $\mu_N X: N^2X \to NX$ is homeomorphic to the projection $\mathrm{pr}_1: I^\tau \times I^\tau \to I^\tau$.

Proof. We carry out the proof only for the implications 1)⇒2) and 2)⇒1). The implications 1)⇒3) and 3)⇒1) are proved similarly.

1)⇒3). Let X be a metrizable continuum. Then by the Moiseev theorem (see E. Moiseev [1988]), $GX \cong Q$. For a proof that the map $\mu_G X$ is homeomorphic to the projection $\mathrm{pr}_1 \colon Q \times Q \to Q$ we use the criterion of Toruńczyk and West [1982]

Fix an $\varepsilon > 0$. Assuming that X is a metric compactum, we equip the spaces GX and G^2X with the Hausdorff metric induced from $\exp^2 X$ and $\exp^2 GX$, respectively.

There exist finite disjoint sets $C_1, C_2 \subset X$ such that the sets $GC_1 \setminus \eta X(C_1)$ and GC_2 are ε-nets in G^2X. For each $B \in \exp X$ let

$$\beta(B) = \bigl\{C \in \exp X \mid C \supset \{c, b\},\ b \in B,\ c \in C_2\bigr\} \in GX.$$

Define $K_1 = (GC_1 \setminus \eta X(C_1)) \cup \{\{X\}\}$, and for each $\mathcal{A} \in GX$ let $K_2(\mathcal{A}) = GC_2 \cup \{\beta(B) \mid B \in \mathcal{A}\}$. It is not hard to see that

$$K_1 \cap K_2(\mathcal{A}) = \varnothing \tag{5.3}$$

and

$$K_1 \cap B^- \neq \varnothing \neq K_2(\mathcal{A}) \cap B^- \tag{5.4}$$

for each $B \in \perp \mathcal{A}$. Direct computations show that the map $K_2 \colon GX \to \exp GX$ is continuous.

We define the maps $h_1, h_2 \colon G^2X \to G^2X$ by

$$\begin{aligned} h_1(\mathfrak{A}) &= \xi_G(\mathfrak{A}, \mu_G X^{-1}(\mu_G X(\mathfrak{A})) \cap X_1^+), \\ h_2(\mathfrak{A}) &= \xi_G(\mathfrak{A}, \mu_G X^{-1}(\mu_G X(\mathfrak{A})) \cap K_2(\mu_G X(\mathfrak{A}))^-). \end{aligned}$$

Since $GC_1 \setminus \eta X(C_1)$ and GC_2 are ε-nets in GX, it follows (Moiseev [1988]) that $d(h_i, \mathrm{id}_{G^2X}) < \varepsilon$, $i = 1, 2$. It follows from (5.3) that

$$h_1(G^X) \cap h_2(G^2X) = \varnothing,$$

and from (5.4) that $\mu_G X \circ h_i = \mu_G X$, $i = 1, 2$. Thus, the map $\mu_G X$ satisfies the disjoint approximation condition. It remains to note that this map is soft (see the proof of Theorem 5.3.1), and to use the Toruńczyk-West theorem.

Suppose that X is a compactum of weight $\tau > \omega$ and $X = \varprojlim S$, where $S = \{X_\alpha, p_{\alpha\beta}; \tau\}$ is a regular system with open projections without points of multiplicity one, and X_0 is a one-point space Denote by p_o

the limit projections of the system $\mathcal{S}$. The map $\mu_G X$ is then the limit of the morphism of the inverse systems $\{\mu_G X_\alpha\}\colon G^2\mathcal{S} \to G\mathcal{S}$. For each $\alpha < \tau$ let $Z_\alpha = G^2 X_\alpha \times_{GX_\alpha} GX$ be the fiberwise product of the maps $\mu_G X_\alpha\colon G^X_\alpha \to GX_\alpha$ and $Gp_\alpha\colon GX \to GX_\alpha$. If $\alpha \geq \beta$, then denote by $t_{\alpha\beta}\colon Z_\alpha \to Z_\beta$ the map given by

$$t_{\alpha\beta}(\mathfrak{A}, \mathcal{A}) = (G^2 p_{\alpha\beta}(\mathfrak{A}), \mathcal{A}), \quad (|GA, \mathcal{A}) \in Z.$$

Then the space G^2X is the inverse limit of the system $\mathcal{S}' = \{Z_\alpha, t_{\alpha\beta}; \tau\}$, and the limit projection $t_0\colon G^2X \to Z_0 = GX$ of the system $\mathcal{S}'$ coincides with the map $\mu_G X$. To prove the theorem it suffices to show (A. Chigogidze [1986], E. Shchepin [1979]) that each projection $t_{al\beta}$, $\alpha > \beta$, is a soft map admitting two disjoint sections.

Let $(\mathfrak{B}, \mathcal{B}) \in Z_\beta$. Then

$$\begin{aligned} t_{\alpha\beta}^{-1}(\mathfrak{B}, \mathcal{B}) = &\{(\mathfrak{A}, \mathcal{B}) \in G^2 X_\alpha \times GX_\alpha \mid \mu X_\alpha(\mathfrak{A}) = Gp_\alpha(\mathcal{B}), \\ &G^2 p_{\alpha\beta}(\mathfrak{A}) = \mathfrak{B}\}. \end{aligned}$$

Setting $\mathcal{B}_\gamma = Gp_\gamma(\mathcal{B})$ for each $\gamma < \tau$, we get

$$\begin{aligned} t_{\alpha\beta}^{-1}(\mathfrak{B}, \mathcal{B}) = &\left(\bigcap\{B^{++} \mid B \in \mathcal{B}_\beta\}\right) \cap \left(\bigcap\{C^{--} \mid C \in \perp\mathcal{B}_\beta\}\right) \\ &\cap \left(\bigcap\{GP_{\alpha\beta}^{-1}(\mathcal{M}^+) \mid \mathcal{M} \in \mathfrak{B}\}\right) \cap \left(\bigcap\{G_{\alpha\beta}^{-1}(\mathcal{N}^-) \mid \mathcal{N} \in \perp\mathfrak{B}\}\right). \end{aligned}$$

From this and the continuity of the operation $\bigcap\colon \exp GK \to GK$ it follows that the map $t_{\alpha\beta}$ is open. Arguing as in the proof of Theorem 5.3.1, we see that $t_{\alpha\beta}$ is soft.

We show that the fibers of $t_{al\beta}$ for $\alpha > \beta$ are not single points. For each $A \in \mathcal{B}_\alpha$ we have that $p_{\alpha\beta}(A) \in \mathcal{B}_\beta = \mu_G X_\beta(\mathfrak{B})$, and hence there exists an $\mathcal{M} \in GB$ such that $p_{\alpha\beta}(A) \in \bigcap \mathcal{M}$. Thus, $p_{\alpha\beta} \in \mathcal{K}$ for any $\mathcal{K} \in \mathcal{M}$. Put

$$\begin{aligned} \mathcal{K}_{A,\mathcal{M}}^1 &= \{C \in \exp X_\alpha \mid C \supset A \text{ or } C \supset p_{\alpha\beta}^{-1}(K) \text{ for some } K \in \mathcal{K}\}, \\ \mathfrak{A}' &= \{\mathcal{A} \in \exp GX_\alpha \mid \mathcal{A} \supset \{\mathcal{K}_{A,\mathcal{M}} \mid \mathcal{K} \in \mathcal{M}\} \text{ for some } \mathcal{M} \in \mathfrak{B}\}. \end{aligned}$$

It can be verified directly that $(\mathfrak{A}', \mathcal{B}) \in t_{\alpha\beta}^{-1}(\mathfrak{B}, \mathcal{B})$.

Consider two cases. 1) All elements of the system $\mathcal{B}_\alpha$ minimal with respect to inclusion are complete inverse images under the map $p_{\alpha\beta}$. Let A be a fixed minimal (with respect to inclusion) element of $\mathcal{B}_\alpha$,

and let $A = A_1 \cup A_2$, where $A_1, A_2 \in \exp X_\alpha$, $A_1 \neq A \neq A_2$, and $p_{\alpha\beta}(A_1) = p_{\alpha\beta}(A_2) = p_{\alpha\beta}(A)$.

2) There exists a minimal (with respect to inclusion) element $A \in \mathcal{B}_\alpha$, $A \neq p_{\alpha\beta}^{-1}(p_{\alpha\beta}(A))$ and $A_1 \cup A_2 = p_{\alpha l\beta}^{-1}(p_{\alpha\beta}(A))$.

In both cases fix $\mathcal{M}_0 \in \mathfrak{B}$ such that $p_{\alpha\beta}(A) \in \bigcap \mathcal{M}_0$, and for each $\mathcal{K} \in C\mathcal{M}_0$ and $i = 1, 2$ put

$$\mathcal{K}_{A_i} = \{C \in \exp X_\alpha \mid C \supset A_i \text{ or } C \supset p_{\alpha\beta}^{-1}(K) \text{ for some } K \in \mathcal{K}\},$$

and $\mathcal{K}'' = \mathcal{K}'_{A_1} \cup \mathcal{K}'_{A_2}$.

Let

$$\mathfrak{A}'' = \mathfrak{A}' \cup \{\mathcal{A} \in \exp GX_\alpha \mid \mathcal{A} \supset \{\mathcal{K}'' \mid CK \in \mathcal{M}_0\}\}.$$

Then $(\mathfrak{A}'', \mathcal{B}) \in t_{\alpha\beta}^{-1}(\mathfrak{B}, \mathcal{B})$, and $\mathfrak{A}' \neq GA''$.

For each $(\mathfrak{B}, \mathcal{B}) \in Z_\beta$ let

$$\Phi(\mathfrak{B}, \mathcal{B}) = \{\mathfrak{A} \in G^2 X_\alpha \mid t_{\alpha\beta}(\mathfrak{A}, \mathcal{B}) = (\mathfrak{B}, \mathcal{B})\},$$

and define the sections ζ_1 and ζ_2 of the map $t_{\alpha\beta}$ by

$$\zeta_1(\mathfrak{B}, \mathcal{B}) = \left(\bigcup \Phi(\mathfrak{B}, \mathcal{B}), \mathcal{B}\right),$$
$$\zeta_2(\mathfrak{B}, \mathcal{B}) = \left(\bigcap \Phi(\mathfrak{B}, \mathcal{B}), \mathcal{B}\right), \quad (\mathfrak{B}, \mathcal{B}) \in Z_\beta.$$

The continuity of ζ_1 and ζ_2 follows from the openness of $t_{\alpha\beta}$ and the continuity of the maps $\bigcup, \bigcap \colon \exp GX_\alpha \to GX_\alpha$; the disjointness of the images of ζ_1 and ζ_2 immediately follows from the fact that the inverse images of the maps $t_{\alpha\beta}$ are not single points.

3)$\Rightarrow$1). If $\mu_G X$ is a soft map, then by Theorem 5.3.12 the continuum X is openly generated. Assume that X is not character homogeneous, and let $x \in X$ be a point of character $\tau' < \tau = \omega(x)$. Let $\mathcal{U}$ be a base at x of cardinality τ'. For any $U \in \mathcal{U}$ let $\widehat{U} = U^+ \cap U^-$. Then $\bigcap\{\widehat{U} \mid U \in \mathcal{U}\} = \{\eta X(x)\}$. Thus, the character of the point $\eta X(x)$ in GX is equal to τ'. By the same considerations, the character of the point $\eta GX \circ \eta X(x)$ in $G^2 X$ is equal to τ', and hence not all the fibers of the map $\mu_G X$ are homeomorphic to I^τ. Contradiction. □

There is also a zero-dimensional counterpart of Theorem 5.3.13, with a similar proof.

Theorem 5.3.14. *For a zero-dimensional compactum X of weight τ the following conditions are equivalent:*

1) X is openly generated and character homogeneous;

2) the map $\mu_G X\colon G^2X \to GX$ is homeomorphic to the projection $\mathrm{pr}_1\colon D^\tau \times D^\tau \to D^\tau$;

3) the map $\mu_N X\colon N^2X \to NX$ is homeomorphic to the $\mathrm{pr}_1\colon D^\tau \times D^\tau \to D^\tau$.

5.3.3. Geometric characterization of power monad

In the sequel, we will need a modification of the operation of tensor product (see Section 3.4). Let $\mathbb{T} = (T, \eta, \mu)$ be a normal monad in **Comp**, $a \in T^2X$, $b \in T^2Y$. Define a map $\hat{f}(b)\colon X \to T^2(X \times Y)$ by the formula $\hat{f}_b(x) = T^2 i_x(b)$, $x \in X$, and let

$$a \widehat{\otimes} b = \mu T(X \times Y) \circ \mu T^2(X \times Y) \circ T^2 \hat{f}_b(a).$$

Lemma 5.3.15. *1) The map $\widehat{\otimes}\colon T^2X \times T^2Y \to T^2(X \times Y)$ is continuous;*

2) the operation $\widehat{\otimes}$ is continuous with respect to both arguments;

3) the operation $\widehat{\otimes}$ is associative;

4) $T^2\mathrm{pr}_1(a\widehat{\otimes}b) = a$, $T^2\mathrm{pr}_2(a\widehat{\otimes}b) = b$;

5) the diagram

$$\begin{tikzcd}
T^2 \times T^2Y \arrow[r, "\widehat{\otimes}"] \arrow[d, "\mu X \times \mu Y"'] & T^2(X \times Y) \arrow[d, "\mu(X \times Y)"] \\
TX \times TY \arrow[r, "\oplus"] & T(X \times Y)
\end{tikzcd}$$

is commutative.

Proof. Properties 1) to 4) can be established similarly to those of Proposition 3.4.2.

5) Let $a \in T^2X$, $b \in T^2Y$, then

$$\begin{aligned}
\mu(X \times Y)(a\widehat{\otimes}b) &= \mu(X \times Y) \circ \mu T(X \times Y) \circ \mu T^2(X \times Y) \circ T^2 \hat{f}_b(a) \\
&= \mu(X \times Y) \circ \mu T(X \times Y) \circ T\mu T(X \times Y) \circ T^2 \hat{f}_b(a) \\
&= \mu(X \times Y) \circ T(\mu(X \times Y) \circ \mu T(X \times Y)) \circ T^2 \hat{f}_b(a) \\
&= \mu(X \times Y) \circ T\mu(X \times Y) \circ T^2(\mu(X \times Y) \circ \hat{f}_b)(a) \\
&= \mu(X \times Y) \circ T\mu(X \times Y) \circ T^2 f_{\mu Y(b)}(a) \\
&= \mu(X \times Y) \circ \mu T(X \times Y) \circ T^2 f_{\mu Y(b)}(a) \\
&= \mu T(X \times Y) \circ T f_{\mu Y(b)} \circ \mu X(a) = \mu X(a) \otimes \mu X(b).
\end{aligned}$$

□

Lemma 5.3.16. *Suppose that* $\mathbb{T} = (T, \eta, \mu)$ *is a normal monad in* **Comp**, $\deg T = \infty$, *and* T *is not a power functor. Then there exists* $c \in TX$, *for some finite* X, *and maps* $f_1, f_2\colon X \to 2$ *satisfying the conditions:*

1) $f_1 \neq f_2$;

2) $Tf_1(c) = Tf_2(c)$;

3) there exists $x_0 \in X$ *such that* $f_1(x_0) = f_2(x_0)$.

Proof. It follows from Theorem 2.9.7 that for some finite Y there exists an element $b \in TY$ and maps $g_0, g_1\colon Y \to Y$ such that $Tg_0(b) = Tg_1(b)$. Fix $d \in T2$ with $\deg(d) = 2$. Let $y_0 \in Y$ be such that $g_1(y_0) \neq g_2(y_0)$. We may suppose that $(y_0, 0) \in \operatorname{supp}(b \otimes d)$. Denote by $\mathcal{R}$ the equivalence relation on $Y \times 2$ with the only nontrivial element $\{(y, 1) \mid y \in Y\}$ and let $q\colon Y \times D \to (Y \times D)/\mathcal{R} = X$ be the quotient map. There exist maps $f_i'\colon X \to X$ such that $f_i' \circ q = q \circ (g_i \times 1_2)$, $i = 1, 2$. Then $f_1' \neq f_2'$ while $Tf_1'(Tq(b\otimes d)) = Tf_2'(Tq(b\otimes d))$. Let $z_0 \in X$ be such that $f_1'(z_0) \neq f_2'(z_0)$ and $r\colon f_1'(X) = f_2'(X) \to \{f_1'(z_0), f_2'(z_0)\}$ be a retraction. Then the element $c = Tq(b \otimes d)$ and the maps $f_i = r \circ f_i'\colon X \to \{f_1'(z_0), f_2'(z_0)\} \cong 2$, $i = 1, 2$, are as required. □

Theorem 5.3.17. *Let* $\mathbb{T} = (T, \eta, \mu)$ *be a normal monad in* **Comp** *such that the map* $\mu I^\tau\colon T^2 I^\tau \to T I^\tau$ *is soft for some* $\tau > \omega_1$. *Then* $\mathbb{T}$ *is a power monad.*

Proof. We may suppose that $\deg T = \infty$, otherwise applying Theorem 3.4.4.

Assume that T is not a power functor. Then there exists $b \in TQ$, $\deg(b) < \infty$, and maps $f_1, f_2 \colon Q \to Q$, $f_1 \neq f_2$, such that $Tf_1(b) = Tf_2(b) = a$. Using Lemma 5.3.16, choose $c \in TQ$, $\deg(c) < \infty$, a point $y_0 \in \operatorname{supp}(c)$ and maps $h_1, h_2 \colon \operatorname{supp}(c) \to 2$ such that $h_1 \neq h_2$, $Th_1(c) = Th_2(c)$ and $h_1(y_0) = h_2(y_0)$, say $h_1(y_0) = h_2(y_0) = 0$.

Since the normal functors preserve weight, the map μI^τ can be embedded into the projection map $\mathrm{pr}_1 : TI^\tau \times I^\tau \to TI^\tau$, and it follows from softness of the map μI^τ that there exists a retraction $r \colon TI^\tau \times I^\tau \to T^2 I^\tau$ making the diagram

$$\begin{tikzcd}
\mathcal{D} = T^2 I^\tau \arrow[dr, "\mu I^\tau"'] & & TI^\tau \times I^\tau \arrow[ll, "r"'] \arrow[dl, "\mathrm{pr}_1"] \\
& TI^\tau &
\end{tikzcd}$$

commute.

Using the fact that $\tau > \omega_1$ and applying the results on spectral analysis of the diagrams for the functors T, T^2 and $T' = T \times \mathrm{Id}$ we conclude that in the category of square diagrams in **Comp** there exists a commutative diagram

$$\begin{tikzcd}
T^2(\pi^3 I^\omega) \arrow[dr, "M"'] & & T'(\pi^3 I^\omega) \arrow[ll, "R"'] \arrow[dl, "\Pi"] \\
& T(\pi^3 I^\omega) &
\end{tikzcd}$$

in which the diagram morphism R consists of retractions, the morphism M is formed by the components of the natural transformation μ, and Π consists of the projections.

Let $x_0 \in I^\omega = Q$. Then the diagram

$$\begin{tikzcd}
\mu(Q^3)^{-1}(\eta Q(x_0) \otimes c \otimes a) \arrow[r, "T^2\pi_{13}"] \arrow[d, "T^2\pi_{12}"'] & \mu(Q^2)^{-1}(\eta Q(x_0) c \otimes a) \arrow[d, "T^2\pi_1"] \\
\mu(Q^2)^{-1}(\eta Q(x_0) \otimes c) \arrow[r, "T^2\pi_1"] & \mu Q^{-1}(\eta Q(x_0))
\end{tikzcd} \tag{5.5}$$

is a retract of the diagram

$$\begin{array}{ccc}
\Pi(Q^3)^{-1}(\eta Q(x_0)\otimes c\otimes a) & \xrightarrow{T'\pi_{13}} & \Pi(Q^2)^{-1}(\eta Q(x_0)c\otimes a) \\
{\scriptstyle T'\pi_{12}}\downarrow & & \downarrow{\scriptstyle T'\pi_1} \\
\Pi(Q^2)^{-1}(\eta Q(x_0)\otimes c) & \xrightarrow{T'\pi_1} & \Pi Q^{-1}(\eta Q(x_0)
\end{array}$$

which is isomorphic to $\pi^3 Q$ and, consequently, is a pullback diagram.

We are going to show that the latter assertion leads to a contradiction. Let

$$K = (\eta TQ\circ\eta Q(x_0))\widehat{\otimes}\eta TQ(c)\widehat{\otimes}T\eta Q(b)$$

and define the maps $g_1, g_2\colon \{x_0\}\times Q^2\to\{x_0\}\times Q^2$ by the formula:

$$g_i(x_0,y,z)=\begin{cases}(x_0,y,f_1(z)), & \text{if } h_1(y)=0,\\ (x_0,y,f_2(z)), & \text{if } h_1(y)=1,\end{cases}$$

$i=1,2$. Put $L_i=T^2g_i(K)$, $i=1,2$, and prove that the following holds:

(a) $\mu Q^3(L_i)=\eta Q(x_0)\otimes c\otimes a$;

(b) $T^2\pi_{1j}(L_1)=T^2\pi_{1j}(L_2)$, $j=2,3$;

(c) $L_1\neq L_2$.

Prove (a). For every $(x,y)\in Q^2$ by $i_{(x,y)}\colon Q\to Q^3$ we denote the map defined by the formula $i_{(x,y)}(z)=(x,y,z)$, $z\in Q$. For every $d\in T^2Q$ let $\hat f_d\colon Q^2\to T^2(Q^3)$ be the map defined by the formula $\hat f_d(x,y)=T^2 i_{(x,y)}(d)$, $(x,y)\in Q^2$.

For every $i=1,2$ there exists $j(i)=1,2$ such that $g_i\circ i_{(x,y)}=i_{(x,y)}\circ f_{j(i)}$. Then

$$\begin{aligned}T^2g_i\circ\hat f_{T\eta Q(b)}(x,y)=&T^2(g_i\circ i_{(x,y)})\circ T\eta Q(b)\\ =T^2 i_{(x,y)}\circ T(Tf_{j(i)}\circ\eta Q)(b)=&T^2 i_{(x,y)}\circ T(\eta Q\circ f_{j(i)})(b)\\ &T^2 i_{(x,y)}\circ T\eta Q(a).\end{aligned}$$

Since

$$\begin{aligned}K=&\eta TQ\circ\eta Q(x_0))\widehat{\otimes}T\eta Q(c)\widehat{\otimes}T\eta Q(b)\\ =&\mu T(Q^3)\circ\mu T^2(Q^3)\circ T^2\hat f_{T\eta Q(b)}((\eta TQ\circ\eta Q(x_0))\widehat{\otimes}\eta TQ(c)),\end{aligned}$$

naturality of μ implies that

$$\begin{aligned}
&T^2 g_i(K)\\
&= \mu T(Q^3) \circ \mu T^2(Q^3) \circ T^2(T^2 g_i \circ \hat{f}_{T\eta Q(b)})((\eta TQ \circ \eta Q(x_0)) \hat{\otimes} \eta TQ(c))\\
&= \mu T(Q^3) \circ \mu T^2(Q^3) \circ T^2 \hat{f}_{T\eta Q(b)}((\eta TQ \circ \eta Q(x_0)) \hat{\otimes} \eta TQ(c))\\
&= ((\eta TQ \circ \eta Q(x_0)) \hat{\otimes} \eta TQ(c) \hat{\otimes} T\eta Q(a),
\end{aligned}$$

$i = 1, 2$. Now we have to apply Lemma 5.3.16.

(b) For every $z \in Q$ denote by $k_z \colon Q \to Q^3$, $\gamma_z \colon \{0,1\} \to Q^2$ the maps acting by the formulae

$$k_z(y) = (x_0, y, z), \ \gamma_z(0) = (x_0, f_1(z)), \ \gamma_z(1) = (x_0, f_2(z)).$$

By $s \colon Q \to TQ^3$ we denote the map $s(z) = Tk_z(c)$, $z \in Q$. Since $T^2\pi_2(K) = \eta TQ(c)$, $T^2\pi_3(K) = T\eta Q(b)$ (recall that π_i is the projection of $K \times K \times K$ onto the i-th coordinate), we see that, by Lemma 5.3.15,

$$\mathrm{Supp}(T^2\pi_2(K)) = \{\mathrm{supp}(c)\}, \ \mathrm{Supp}(T^2\pi_3(K)) = \{\{z\} \mid z \in \mathrm{supp}(b)\},$$

whence, by preimage-preserving property, $K = Ts(b)$.

Note that

$$\pi_{13} \circ g_i \circ k_z = \gamma_z \circ h_i, \ i = 1, 2. \tag{5.6}$$

Since

$$T_2\pi_{13}(L_i) = T^2(\pi_{13} \circ g_i)(K) = T(T(\pi_{13} \circ g_i) \circ s)(b),$$

taking into account (5.6) we obtain

$$\begin{aligned}
T(\pi_{13} \circ g_1) \circ s(z) &= T(\pi_{13} \circ g_1 \circ k_z)(c) = T(\gamma_z \circ h_1)(c)\\
&= T(\gamma_z \circ h_2)(c) = T(\pi_{13} \circ g_2) \circ s(z),
\end{aligned}$$

whence

$$\begin{aligned}
T^2\pi_{13}(L_1) &= T^2(\pi_{13} \circ g_1)(K) = T(T(\pi_{13} \circ g_1) \circ s)(b)\\
&= T(T(\pi_{13} \circ g_1) \circ s)(b) = T^2\pi_{13}(L_2).
\end{aligned}$$

The equality $T^2\pi_{12}(L_1) = T^2\pi_{12}(L_2)$ is obvious.

(c) Suppose that $y_1 \in \mathrm{supp}(c)$ is a point such that $h_1(y_1) \neq h_2(y_1)$, e. g. $h_1(y_1) = 0$, $h_2(y_1) = 1$. It is not difficult to see that for every

$A \in \mathrm{Supp}(L_1)$ the set $\pi_{13}(A \cap (\{x_0\} \times \{y_0, y_1\} \times Q))$ is a singleton; at the same time there exists $B \in \mathrm{Supp}(L_2)$ such that $|\pi_{13}(B \cap (\{x_0\} \times \{y_0, y_1\} \times Q))| = 2$. Thus, $L_1 \neq L_2$.

Properties (a)–(c) imply that diagram (5.5) is not a pullback diagram. □

The proof of the following theorem is similar. We have only to remark that the finite normal functors preserve the class of zero-dimensional compact Hausdorff spaces.

Theorem 5.3.18. *Let* $\mathbb{T} = (T, \eta, \mu)$ *be a normal monad in* **Comp** *such that* T *is a finite functor and the map* $\mu 2^\tau \colon T^2 2^\tau \to T 2^\tau$ *is 0-soft for some* $\tau > \omega_1$. *Then* $T \cong (-)^n$ *for some* $n \in \mathbb{N}$.

5.4. Notes and comments to Chapter 5

The method of characteristics in investigations of uncountable functor-powers (Theorem 5.1.4) is invented by E. Shchepin [1981]. Theorems 5.1.7–5.1.7 are proved in M. Smurov [1980], and Theorem 5.1.30 in M. Smurov [1983]. Theorem 5.2.3 is proved by A. Ivanov [1981]. The examples of this Section are constructed by M. Zarichnyi [1990a].

The notion of openly generated space is introduced by E. Shchepin [1975]. Theorems 5.3.1, 5.3.2, 5.3.7, and 5.3.8 are due to M. Zarichnyi [1989] and [1991c].

For Theorems 5.3.12, 5.3.13, and 5.3.14 see M. Zarichnyi [1991c].

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